(1) Consider the set of points \{(0, 2), (1, 3), (2, 10)\}.

(a) Find the three quadratic Lagrange interpolating polynomials \(L_0, L_1,\) and \(L_2\) for this set of points.

\[
L_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2} (x^2 - 3x + 2)
\]

\[
L_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-2)}{(-1)(-1)} = -(x^2 - 2x)
\]

\[
L_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x-1)}{2(1)} = \frac{1}{2} (x^2 - x)
\]

(b) Find the unique quadratic polynomial \(P_2(x)\) through these points using the Lagrange interpolation formulation. Simplify your answer to the form \(ax^2 + bx + c\) where \(a, b,\) and \(c\) are exact, rational numbers.

\[
P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)
\]

\[
= 2(\frac{1}{2}) (x^2 - 3x + 2) - 3(x^2 - 2x) + 10(\frac{1}{2}) (x^2 - x)
\]

\[
= x^2 - 3x + 2 - 3x^2 + 6x + 5x^2 - 5x
\]

\[
= 3x^2 - 2x + 2
\]

\[
P_2(x) = 3x^2 - 2x + 2
\]
(2) Let \( f(x) = \sin^{-1} x \) and let \( x_0 = 0 \), \( x_1 = 0.1 \) and \( x_2 = 0.2 \). Determine the divided differences

\[ f[x_0, x_1], \quad \text{and} \quad f[x_0, x_1, x_2]. \]

Express your answers with 5 digits to the right of the decimal. Do not give answers in scientific notation.

\[
f[x_0, x_1] = f[0, 0.1] = \frac{\sin^{-1}(0.1) - \sin^{-1}(0)}{0.1 - 0} = 1.00167
\]

\[
f[x_1, x_2] = f[0.1, 0.2] = \frac{\sin^{-1}(0.2) - \sin^{-1}(0.1)}{0.2 - 0.1} = 1.01190
\]

\[
f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.01190 - 1.00167}{0.2} = 0.05115
\]

\[
f[x_0, x_1] = 1.00167
\]

\[
f[x_0, x_1, x_2] = 0.05115
\]
(3) Suppose we use $P_2(x)$ to interpolate the function $f(x) = e^{x/3}$ on the interval $-1 \leq x \leq 1$.

(a) Determine the nodes $\{x_0, x_1, x_2\}$ that will minimize the error. Give the values exact or with 5 digits to the right of the decimal. Do not give answers in scientific notation.

Choose the roots of $T_3$, the Chebyshev polynomial of degree 3.

$$x_j = \cos \left( \frac{2j+1}{6} \pi \right) \quad j = 0, 1, 2$$

$$x_0 = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} \approx 0.86603$$
$$x_1 = \cos \left( \frac{\pi}{3} \right) = 0$$
$$x_2 = \cos \left( \frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2} \approx -0.86603$$

(b) Find the best bound on the error $|f(x) - P_2(x)|$ over the interval $[-1, 1]$. Give your answer with 5 digits to the right of the decimal. Do not give answers in scientific notation.

$$f'(x) = \frac{1}{3} e^{x/3}, \quad f''(x) = \frac{1}{9} e^{x/3}, \quad f'''(x) = \frac{1}{27} e^{x/3}$$

$$\left| \frac{f^{(3)}(x)}{3!} \right| \leq \frac{1}{27} e^{1/3}$$

and

$$\left| f(x) - P_2(x) \right| \leq \frac{1}{27} e^{1/3}$$

$$\left| f(x) - P_2(x) \right| \leq \frac{1}{27} e^{1/3} \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} \left| f'''(c) \right|$$

$$\leq \frac{1}{27} e^{1/3} \frac{(\frac{1}{2} e^{1/3})}{6} \leq 0.00215$$
(4) We wish to find the natural cubic spline interpolating the points \{(1, 1), (2, 0), (3, 3)\}.

(a) Find the three coefficients \(M_1\), \(M_2\), and \(M_3\).

\[ M_1 = M_3 = 0 \quad , \quad h_j = 1 \quad \text{for} \quad j = 1, 2 \]

\[ \frac{1}{6} M_1 + \frac{2}{3} M_2 + \frac{1}{6} M_3 = y_3 - 2y_2 + y_1 \]

\[ \frac{2}{3} M_2 = 3 - 0 + 1 = 4 \]

\[ M_2 = 4 \left( \frac{3}{2} \right) = 6 \]

\[ \begin{align*}
M_1 &= 0 \\
M_2 &= 6 \\
M_3 &= 0
\end{align*} \]
The points for this problem are \(((1, 1), (2, 0), (3, 3))\).

(b) Determine the part of the spline valid for \(1 \leq x \leq 2\). Simplify to the form \(ax^3 + bx^2 + cx + d\) where \(a, b, c,\) and \(d\) are given as exact rational numbers.

\[
M_{z} = 6 \quad \quad S(x) = \\
- \frac{M_{1}^{0}}{6} (x - x_1)^3 + \frac{M_{2}^{0}}{6} (x - x_1)^3 + q_{1} (x - x_1) + y_{z}^{0} (x - x_1)
\]

\[
= \frac{-1}{6} \left[ M_{1}^{0} (x_2 - x) + M_{2}^{0} (x - x_1) \right]
\]

\[
= (x - 1)^3 + (2 - x) - (x - 1)
\]

\[
= x^3 - 3x^2 + 3x - 1 + 2 - x - x + 1
\]

\[
= x^3 - 3x^2 + x + 2
\]

for \(1 \leq x \leq 2\)
The points for this problem are \( \{(1, 1), (2, 0), (3, 3)\} \).

\[ s(x) = \begin{cases} \frac{M_2}{6} (x_3 - x)^3 + \frac{M_3}{6} (x - x_2)^3 + \frac{y_2}{6} (x_3 - x) + \frac{y_3}{6} (x - x_2) \\ \frac{1}{6} \left[ M_2 (x_3 - x) + M_3 (x - x_2) \right] \end{cases} \]

\[ = (3 - x)^3 + 3 (x - 2) - (3 - x) \]

\[ = -x^3 + 9x^2 - 27x + 27 + 3x - 6 - 3 + x \]

\[ = -x^3 + 9x^2 - 23x + 18 \]

Express the final answer in the space provided:

\[ s(x) = \begin{cases} x^3 - 3x^2 + x + 2 & , \quad 1 \leq x \leq 2 \\ -x^3 + 9x^2 - 23x + 18 & , \quad 2 \leq x \leq 3 \end{cases} \]
(5) Consider the integral
\[ \int_0^4 \frac{dx}{x^3 + 1}. \]

(a) Approximate the integral using \( T_4 \). Give your answer with 5 digits to the right of the decimal place. Do not give your answer in scientific notation. To receive credit, show the intermediate steps.

\[ h = \frac{4-0}{4} = 1 \]

\[ x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 3 \quad x_4 = 4 \]

\[ f(x_0) = 1 \quad f(x_1) = \frac{1}{2} \quad f(x_2) = \frac{1}{4} \]

\[ f(x_3) = \frac{1}{8} \quad f(x_4) = \frac{1}{16} \]

\[ T_4 = \frac{b-a}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right] \]

\[ = \frac{1}{4} \left[ 1 + 1 + \frac{3}{4} + \frac{3}{8} + \frac{1}{16} \right] \]

\[ = 1.15452 \]
(b) Approximate the integral using $S_4$. Give your answer with 5 digits to the right of the decimal place. Do not give your answer in scientific notation. To receive credit, show the intermediate steps.

The $h$, $x_j$ and $f(x_j)$ values are from part (a).

$$S_4 = \frac{h}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right]$$

$$= \frac{1}{3} \left[ 1 + 2 + \frac{2}{9} + \frac{4}{36} + \frac{1}{66} \right]$$

$$= 1.12682$$