CENTER OF PLANAR QUINTIC QUASI–HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we first characterize all quasi–homogeneous but non–homogeneous planar polynomial differential systems of degree five, and then among which we classify all the ones having a center at the origin. Finally we present the topological phase portrait of the systems having a center at the origin.

1. Introduction

The problem on the classification of polynomial differential systems having a center has been intensively studied. Bautin [5] completed the classification of the center–focus problem for quadratic differential systems. Some further studies can be found in [15, 25, 28, 29, 31] and the references therein. Malkin [19] and Vulpe and Sibirskii [27] classified the centers of cubic polynomial systems formed by linear plus homogeneous nonlinearities of degree three. But the center–focus problem for general cubic polynomial differential systems remains open. Any way, for polynomial differential systems of degree larger than two, there are richer partial results on the center–focus problem. For more information, see for instance [8, 22, 23, 28, 29, 31, 32] and the references therein.

Consider a real planar polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$ and the origin $O = (0, 0)$ is a singularity of system (1). As usual, $\mathbb{R}[x, y]$ denotes the ring of polynomials in the variables x and y with coefficients in $\mathbb{R}$, and the dot denotes derivative with respect to an independent variable t. We say that system (1) has *degree* n if $n = \max\{\deg P, \deg Q\}$. System (1) is *reducible* if the polynomials $P(x, y)$ and $Q(x, y)$ have a non–constant common factor in the ring $\mathbb{R}[x, y]$. Otherwise, we will say that system (1) is *coprime*. In what follows we denote by $\mathfrak{X}$ the vector field associated to system (1).

System (1) is called a *quasi–homogeneous polynomial differential system* if there exist constants $s_1, s_2, d \in \mathbb{N}$ such that for an arbitrary $\gamma \in \mathbb{R}^+$ it

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holds

$$P(\gamma^{s_1}x, \gamma^{s_2}y) = \gamma^{s_1+d-1}P(x,y) \quad \text{and} \quad Q(\gamma^{s_1}x, \gamma^{s_2}y) = \gamma^{s_2+d-1}Q(x,y),$$

where $\mathbb{R}^+$ is the set of positive numbers. We call (s_1, s_2) *weight exponents* of system (1) and d *weight degree* with respect to the weight exponents. Moreover, we call $w = (s_1, s_2, d)$ *weight vector* of system (1) or of vector field $\mathfrak{X}$. For a quasi–homogeneous polynomial differential system (1), a weight vector $\widetilde{w} = (\widetilde{s}_1, \widetilde{s}_2, \widetilde{d})$ is *minimal* for system (1) if any other weight vector (s_1, s_2, d) of system (1) satisfies $\tilde{s}_1 \le s_1, \tilde{s}_2 \le s_2$ and $\tilde{d} \le d$. Clearly each quasi–homogeneous polynomial differential system has a unique minimal weight vector. When $s_1 = s_2 = 1$, system (1) is a homogeneous one of degree d.

Quasi–homogeneous polynomial differential systems have also been intensively investigated by many authors from integrability point of view, see for instance [2, 12, 13, 14]. In fact, all planar quasi–homogeneous vector fields are Liouvillian integrable, see e.g. [9, 10, 16]. Specially, for the polynomial and rational integrability of planar quasi–homogeneous vector fields we refer to [3, 6, 18, 30], and for the centers and limit cycles problems we refer to [1, 11, 16] and the references therein.

Recently, García *et al* [10] provided an algorithm to compute quasi–homogeneous but non–homogeneous polynomial differential systems with a given degree. As an application of their algorithm they obtained all the quadratic and cubic quasi–homogeneous but non–homogeneous vector fields. Aziz *et al* [4] characterized all cubic quasi–homogeneous polynomial differential equations which have a center, and proved that such systems do not admit isochronous centers. Liang, Huang and Zhao [17] classified all quartic quasi–homogeneous but non–homogeneous polynomial differential systems and provided their topological phase portraits.

This paper has two aims. The first is to complete the classification of all planar quintic quasi–homogeneous but non–homogeneous polynomial differential systems. The second is to provide a solution to the center–focus problem of these kinds of systems and their phase portraits.

Our first result classifies the quintic quasi–homogeneous but non–homogeneous polynomial vector fields.

Theorem 1. *Every planar real quintic quasi–homogeneous but non–homogeneous coprime polynomial differential system* (1) *can be written as one of the following 15 systems*

$$X_{011}: \quad \dot{x} = a_{05}y^5 + a_{13}xy^3 + a_{21}x^2y, \ \dot{y} = b_{04}y^4 + b_{12}xy^2 + b_{20}x^2,$$
with $a_{05}b_{20} \neq 0$ *and the weight vector* $\widetilde{w} = (2, 1, 4)$,

$$X_{012}: \quad \dot{x} = a_{05}y^5 + a_{22}x^2y^2, \ \dot{y} = b_{13}xy^3 + b_{30}x^3,$$
with $a_{05}b_{30} \neq 0$ *and the weight vector* $\widetilde{w} = (3, 2, 8)$,

$$X_{014}: \quad \dot{x} = a_{05}y^5 + a_{40}x^4, \ \dot{y} = b_{31}x^3y,$$
with $a_{05}a_{40}b_{31} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 4, 16)$,

$X_{015}: \quad \dot{x} = a_{05}y^5, \ \dot{y} = b_{40}x^4,$
with $a_{05}b_{40} \neq 0$ *and the weight vector* $\widetilde{w} = (6, 5, 20)$,
$X_{021}: \quad \dot{x} = a_{05}y^5 + a_{12}xy^2, \ \dot{y} = b_{03}y^3 + b_{10}x,$
with $a_{05}b_{10} \neq 0$ *and the weight vector* $\widetilde{w} = (3, 1, 3)$,
$X_{023}: \quad \dot{x} = a_{05}y^5 + a_{30}x^3, \ \dot{y} = b_{21}x^2y,$
with $a_{05}a_{30}b_{21} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 3, 11)$,
$X_{032}: \quad \dot{x} = a_{05}y^5 + a_{20}x^2, \ \dot{y} = b_{11}xy,$
with $a_{05}a_{20}b_{11} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 2, 6)$,
$X_{111}: \quad \dot{x} = a_{14}xy^4 + a_{22}x^2y^2 + a_{30}x^3, \ \dot{y} = b_{05}y^5 + b_{13}xy^3 + b_{21}x^2y,$
with $a_{30}b_{05} \neq 0$ *and the weight vector* $\widetilde{w} = (2, 1, 5)$,
$X_{113}: \quad \dot{x} = a_{14}xy^4 + a_{40}x^4, \ \dot{y} = b_{05}y^5 + b_{31}x^3y,$
with $a_{40}b_{05} \neq 0$ *and the weight vector* $\widetilde{w} = (4, 3, 13)$,
$X_{114}: \quad \dot{x} = a_{14}xy^4, \ \dot{y} = b_{05}y^5 + b_{40}x^4,$
with $a_{14}b_{05}b_{40} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 4, 17)$,
$X_{123}: \quad \dot{x} = a_{14}xy^4, \ \dot{y} = b_{05}y^5 + b_{30}x^3,$
with $a_{14}b_{05}b_{30} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 3, 13)$,
$X_{131}: \quad \dot{x} = a_{14}xy^4 + a_{20}x^2, \ \dot{y} = b_{05}y^5 + b_{11}xy,$
with $a_{20}b_{05} \neq 0$ *and the weight vector* $\widetilde{w} = (4, 1, 5)$,
$X_{132}: \quad \dot{x} = a_{14}xy^4, \ \dot{y} = b_{05}y^5 + b_{20}x^2,$
with $a_{14}b_{05}b_{20} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 2, 9)$,
$X_{141}: \quad \dot{x} = a_{14}xy^4, \ \dot{y} = b_{05}y^5 + b_{10}x,$
with $a_{14}b_{05}b_{10} \neq 0$ *and the weight vector* $\widetilde{w} = (5, 1, 5)$,
$X_{1}: \quad \dot{x} = a_{05}y^5 + a_{10}x, \ \dot{y} = b_{01}y,$
with $a_{05}a_{10}b_{01} \neq 0$, *and the weight vector* $\widetilde{w} = (5, 1, 1)$.

Our second result completes the characterization of planar quintic quasi–homogeneous but non–homogeneous polynomial vector fields which have a center at the origin.

Theorem 2. *The quintic quasi–homogeneous but non–homogeneous coprime polynomial differential system* (1) *having a center at the origin, together with possible rescalings of variables, must be of the form*

$$\dot{x} = axy^2 - y^5, \qquad \dot{y} = by^3 + x, \tag{2}$$

with $a = -3b$ *and* $b^2 < \frac{1}{3}$. *Furthermore, the center is not isochronous and the period of the periodic orbits is a monotonic function.*

We remark that this last theorem solves the center–focus problem for quintic quasi–homogeneous but non–homogeneous polynomial differential systems. But the problem is still open for quintic homogeneous polynomial differential systems.

The last one presents the phase portrait of system (2) having a center at the origin.

Theorem 3. *The center of system* (2) *under the conditions of Theorem 2 is global, and consequently the topological phase portrait is Figure 1.*

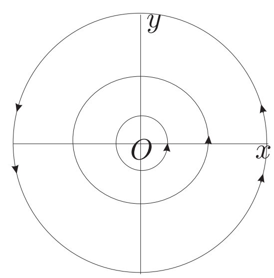

Figure 1. Global phase portrait of system (2).

This paper is organized as follows. Section 2 will prove Theorem 1. Section 3 is devoted to prove Theorem 2, where we mainly check if there exist orbits connecting the origin by the method of generalized normal sectors. Finally we prove Theorem 3 by using the Poincaré compactification.

2. Classification of quintic quasi–homogeneous vector fields

The main aim of this section is to prove Theorem 1. For doing so, we need to compute all quintic quasi–homogeneous but non–homogeneous polynomial differential systems with the minimal weight vector $w = (s_1, s_2, d)$. Without loss of generality we assume that $s_1 > s_2$, otherwise we can exchange the coordinates x and y.

By [10, Proposition 10], if system (1) is quasi–homogeneous but non–homogeneous of degree n with the weight vector (s_1, s_2, d) and $d > 1$, then the system has the minimal weight vector

$$\widetilde{w} = \left(\frac{t+k}{s},\ \frac{k}{s},\ 1 + \frac{(p-1)t + (n-1)k}{s}\right),$$

with $t \in \{1, 2, ..., n-p\}$, $p \in \{0, 1, ..., n-1\}$, $k \in \{1, \ldots, n-p-t+1\}$ satisfying

$$s_1 = \frac{(t+k)(d-1)}{D}, \qquad s_2 = \frac{k(d-1)}{D},$$

where $D = (p-1)t + (n-1)k$, and $s = \gcd(t, k)$. Furthermore, by the algorithm posed in subsection 3.1 of [10] we get that the quasi–homogeneous but non–homogeneous polynomial differential system (1) of degree n with the weight vector (s_1, s_2, d) can be written in

$$X_{ptk} = X_n^p + X_{n-t}^{ptk} + \sum_{\substack{s \in \{1,\dots,n-p\}\setminus\{t\} \\ k_s t = ks \text{ and} \\ k_s \in \{1,\dots,n-s-p+1\}}} X_{n-s}^{psk_s}, \tag{3}$$

where

$$X_n^p = (a_{p,n-p}x^p y^{n-p}, b_{p-1,n-p+1}x^{p-1}y^{n-p+1}),$$

is the homogeneous part of degree n with coefficients not simultaneous vanishing, and

$$X_{n-t}^{ptk} = (a_{p+k,n-t-p-k}x^{p+k}y^{n-t-p-k},\ b_{p+k-1,n-t-p-k+1}x^{p+k-1}y^{n-t-p-k+1}).$$

In order for X_{ptk} to be quasi–homogeneous but non–homogeneous of degree n we must have $X_n^p \not\equiv 0$ and at least one of the other elements not identically vanishing.

Using this last algorithm and the associated notations, we can prove the next result.

Lemma 4. *A quasi–homogeneous but non–homogeneous quintic polynomial differential system with the minimal weight vector $\widetilde{w} = (s_1, s_2, d)$ and $s_1 > s_2$ belongs to the set of following systems:*

$X_{011}:$ $\dot{x} = a_{05}y^5 + a_{13}xy^3 + a_{21}x^2y,\ \dot{y} = b_{04}y^4 + b_{12}xy^2 + b_{20}x^2,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (2,1,4),$

$X_{012}:$ $\dot{x} = a_{05}y^5 + a_{22}x^2y^2,\ \dot{y} = b_{13}xy^3 + b_{30}x^3,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (3,2,8),$

$X_{013}:$ $\dot{x} = a_{05}y^5 + a_{31}x^3y,\ \dot{y} = b_{22}x^2y^2,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (4,3,12),$

$X_{014}:$ $\dot{x} = a_{05}y^5 + a_{40}x^4,\ \dot{y} = b_{31}x^3y,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (5,4,16),$

$X_{015}:$ $\dot{x} = a_{05}y^5,\ \dot{y} = b_{40}x^4,$
with $a_{05}b_{40} \neq 0$ *and* $\widetilde{w} = (6,5,20),$

$X_{021}:$ $\dot{x} = a_{05}y^5 + a_{12}xy^2,\ \dot{y} = b_{03}y^3 + b_{10}x,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (3,1,3),$

$X_{023}:$ $\dot{x} = a_{05}y^5 + a_{30}x^3,\ \dot{y} = b_{21}x^2y,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (5,3,11),$

$X_{031}:$ $\dot{x} = a_{05}y^5 + a_{11}xy,\ \dot{y} = b_{02}y^2,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (4,1,2),$

$X_{032}:$ $\dot{x} = a_{05}y^5 + a_{20}x^2,\ \dot{y} = b_{11}xy,$
with $a_{05} \neq 0$ *and* $\widetilde{w} = (5, 2, 6),$

$X_{111}:$ $\dot{x} = a_{14}xy^4 + a_{22}x^2y^2 + a_{30}x^3,\ \dot{y} = b_{05}y^5 + b_{13}xy^3 + b_{21}x^2y,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (2, 1, 5),$

$X_{112}:$ $\dot{x} = a_{14}xy^4 + a_{31}x^3y,\ \dot{y} = b_{05}y^5 + b_{22}x^2y^2,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (3, 2, 9),$

$X_{113}:$ $\dot{x} = a_{14}xy^4 + a_{40}x^4,\ \dot{y} = b_{05}y^5 + b_{31}x^3y,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (4, 3, 13),$

$X_{114}:$ $\dot{x} = a_{14}xy^4,\ \dot{y} = b_{05}y^5 + b_{40}x^4,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (5, 4, 17),$

$X_{121}:$ $\dot{x} = a_{14}xy^4 + a_{21}x^2y,\ \dot{y} = b_{05}y^5 + b_{12}xy^2,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (3, 1, 5),$

$X_{123}:$ $\dot{x} = a_{14}xy^4,\ \dot{y} = b_{05}y^5 + b_{30}x^3,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (5, 3, 13),$

$X_{131}:$ $\dot{x} = a_{14}xy^4 + a_{20}x^2,\ \dot{y} = b_{05}y^5 + b_{11}xy,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (4, 1, 5),$

$X_{132}:$ $\dot{x} = a_{14}xy^4,\ \dot{y} = b_{05}y^5 + b_{20}x^2,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (5, 2, 9),$

$X_{141}:$ $\dot{x} = a_{14}xy^4,\ \dot{y} = b_{05}y^5 + b_{10}x,$
with $a_{14}^2 + b_{05}^2 \neq 0$ *and* $\widetilde{w} = (5, 1, 5),$

$X_{211}:$ $\dot{x} = a_{23}x^2y^3 + a_{31}x^3y,\ \dot{y} = b_{14}xy^4 + b_{22}x^2y^2 + b_{30}x^3,$
with $a_{23}^2 + b_{14}^2 \neq 0$ *and* $\widetilde{w} = (2, 1, 6),$

$X_{212}:$ $\dot{x} = a_{23}x^2y^3 + a_{40}x^4,\ \dot{y} = b_{14}xy^4 + b_{31}x^3y,$
with $a_{23}^2 + b_{14}^2 \neq 0$ *and* $\widetilde{w} = (3, 2, 10),$

$X_{213}:$ $\dot{x} = a_{23}x^2y^3,\ \dot{y} = b_{14}xy^4 + b_{40}x^4,$
with $a_{23}^2 + b_{14}^2 \neq 0$ *and* $\widetilde{w} = (4, 3, 14),$

$X_{221}:$ $\dot{x} = a_{23}x^2y^3 + a_{30}x^3,\ \dot{y} = b_{14}xy^4 + b_{21}x^2y,$
with $a_{23}^2 + b_{14}^2 \neq 0$ *and* $\widetilde{w} = (3, 1, 7),$

$X_{231}:$ $\dot{x} = a_{23}x^2y^3,\ \dot{y} = b_{14}xy^4 + b_{20}x^2,$
with $a_{23}^2 + b_{14}^2 \neq 0$ *and* $\widetilde{w} = (4, 1, 8),$

X_{311} $\dot{x} = a_{32}x^3y^2 + a_{40}x^4,\ \dot{y} = b_{23}x^2y^3 + b_{31}x^3y,$
with $a_{32}^2 + b_{23}^2 \neq 0$ *and* $\widetilde{w} = (2, 1, 7),$

X_{312} $\dot{x} = a_{32}x^3y^2,\ \dot{y} = b_{23}x^2y^3 + b_{40}x^4,$
with $a_{32}^2 + b_{23}^2 \neq 0,$ *and* $\widetilde{w} = (3, 2, 11),$

X_{321} $\dot{x} = a_{32}x^3y^2,\ \dot{y} = b_{23}x^2y^3 + b_{30}x^3,$
with $a_{32}^2 + b_{23}^2 \neq 0,$ *and* $\widetilde{w} = (3, 1, 9),$

X_{411} $\dot{x} = a_{41}x^4y,\ \dot{y} = b_{32}x^3y^2 + b_{40}x^4,$
with $a_{41}^2 + b_{32}^2 \neq 0,$ *and* $\widetilde{w} = (2, 1, 8),$

$$X_1: \quad \dot{x} = a_{05}y^5 + a_{10}x, \ \dot{y} = b_{01}y, \quad \textit{with } a_{05} \neq 0, \ \textit{and } \widetilde{w} = (5,1,1),$$

where the coefficients a_{ij} and b_{ij} with $i+j \le 4$ in each system are not all equal to zeros.

Proof. According to the above mentioned algorithm and (3), we compute the quasi–homogeneous but non–homogeneous quintic polynomial differential systems with the minimal weight vector $\widetilde{w} = (s_1, s_2, d)$, where $s_1 > s_2$ and $d > 1$. Here p appearing in X_{ptk} of the algorithm has five choices: $p = 0, 1, 2, 3, 4$.

Case 1 $p = 0$. Direct calculations show that

$$\begin{aligned}
&X_{011} = X_5^0 + X_4^{011} + X_3^{022} + X_2^{033}, && \textit{with } \widetilde{w} = (2,1,4),\\
&X_{012} = X_5^0 + X_4^{012} + X_3^{024}, && \textit{with } \widetilde{w} = (3,2,8),\\
&X_{013} = X_5^0 + X_4^{013}, && \textit{with } \widetilde{w} = (4,3,12),\\
&X_{014} = X_5^0 + X_4^{014}, && \textit{with } \widetilde{w} = (5,4,16),\\
&X_{015} = X_5^0 + X_4^{015}, && \textit{with } \widetilde{w} = (6,5,20),\\
&X_{021} = X_5^0 + X_3^{021} + X_2^{042}, && \textit{with } \widetilde{w} = (3,1,3),\\
&X_{022} = X_{011}, && \textit{with } \widetilde{w} = (2,1,4),\\
&X_{023} = X_5^0 + X_3^{023}, && \textit{with } \widetilde{w} = (5,3,11),\\
&X_{024} = X_{012}, && \textit{with } \widetilde{w} = (3,2,8),\\
&X_{031} = X_5^0 + X_2^{031}, && \textit{with } \widetilde{w} = (4,1,2),\\
&X_{032} = X_5^0 + X_2^{032}, && \textit{with } \widetilde{w} = (5,2,6),\\
&X_{033} = X_{011}, && \textit{with } \widetilde{w} = (2,1,4),\\
&X_{042} = X_{021}, && \textit{with } \widetilde{w} = (3,1,3).
\end{aligned}$$

Case 2 $p = 1$. We have

$$\begin{aligned}
&X_{111} = X_5^1 + X_4^{111} + X_3^{122}, && \textit{with } \widetilde{w} = (2,1,5),\\
&X_{112} = X_5^1 + X_4^{112}, && \textit{with } \widetilde{w} = (3,2,9),\\
&X_{113} = X_5^1 + X_4^{113}, && \textit{with } \widetilde{w} = (4,3,13),\\
&X_{114} = X_5^1 + X_4^{114}, && \textit{with } \widetilde{w} = (5,4,17),\\
&X_{121} = X_5^1 + X_3^{121}, && \textit{with } \widetilde{w} = (3,1,5),\\
&X_{122} = X_{111}, && \textit{with } \widetilde{w} = (2,1,5),\\
&X_{123} = X_5^1 + X_3^{123}, && \textit{with } \widetilde{w} = (5,3,13),
\end{aligned}$$

$$X_{131} = X_5^1 + X_2^{131}, \qquad \textit{with } \widetilde{w} = (4,1,5),$$
$$X_{132} = X_5^1 + X_2^{132}, \qquad \textit{with } \widetilde{w} = (5,2,9),$$
$$X_{141} = X_5^1 + X_1^{141}, \qquad \textit{with } \widetilde{w} = (5,1,5),$$

Case 3 $p = 2$. We have

$$X_{211} = X_5^2 + X_4^{211} + X_3^{222}, \qquad \textit{with } \widetilde{w} = (2,1,6),$$
$$X_{212} = X_5^2 + X_4^{212}, \qquad \textit{with } \widetilde{w} = (3,2,10),$$
$$X_{213} = X_5^2 + X_4^{213}, \qquad \textit{with } \widetilde{w} = (4,3,14),$$
$$X_{221} = X_5^2 + X_3^{221}, \qquad \textit{with } \widetilde{w} = (3,1,7),$$
$$X_{222} = X_{211}, \qquad \textit{with } \widetilde{w} = (2,1,6),$$
$$X_{231} = X_5^2 + X_2^{231}, \qquad \textit{with } \widetilde{w} = (4,1,8),$$

Case 4 $p = 3$. We have

$$X_{311} = X_5^3 + X_4^{311}, \qquad \textit{with } \widetilde{w} = (2,1,7),$$
$$X_{312} = X_5^3 + X_4^{312}, \qquad \textit{with } \widetilde{w} = (3,2,11),$$
$$X_{321} = X_5^3 + X_3^{321}, \qquad \textit{with } \widetilde{w} = (3,1,9),$$

Case 5 $p = 4$. We have

$$X_{411} = X_5^4 + X_4^{411}, \qquad \textit{with } \widetilde{w} = (2,1,8).$$

Then Lemma 4 for $d > 1$ follows from these last expressions, and the conditions on the coefficients a_{ij} and b_{ij} in each system of Lemma 4 hold because the systems are of degree five and non–homogeneous.

When $d = 1$, by [10, Proposition 9] the possible quasi–homogeneous but non–homogeneous quintic polynomial system is the system X_1 of Lemma 4 with the coefficients a_{05}, a_{10} and b_{01} non–zero. It is easy to check that the minimal weight vector of the system $\mathcal{X}_1$ is $(5,1,1)$. We complete the proof of the lemma. □

Proof of Theorem 1. By Lemma 4, there are 28 quasi–homogeneous but non–homogeneous quintic polynomial systems with the minimal weight vector $\widetilde{w} = (s_1, s_2, d)$, $s_1 > s_2$ and $d \geq 1$. We can check that the vector fields X_{013}, X_{031}, X_{112} and X_{121} have the common factor y, whereas the vector fields X_{211}, X_{212}, X_{213}, X_{221}, X_{231}, X_{311}, X_{312}, X_{321} and X_{411} have the common factor x. These vector fields can be treated as quasi–homogeneous ones of degree less than 5. Hence the set of quasi–homogeneous but non–homogeneous quintic coprime polynomial differential system is that as listed in Theorem 1. We complete the proof of the theorem. □

3. Proof of Theorem 2

From Theorem 1, there exist 15 quasi–homogeneous but non–homogeneous coprime quintic polynomial differential systems. We note that the vector fields X_{014}, X_{023}, X_{032}, X_{111}, X_{113}, X_{131} and X_1 have the invariant line $y = 0$, and that the vector fields X_{111}, X_{113}, X_{114}, X_{123}, X_{131}, X_{132} and X_{141} have the invariant line $x = 0$. So their origin cannot be a center. There remain only four vector fields X_{011}, X_{012}, X_{015} and X_{021} to be studied.

We first consider X_{015}. Observe that it is a Hamiltonian system and its origin is a degenerate singularity.

Lemma 5. *The origin O of the Hamiltonian system X_{015} consists of two hyperbolic sectors.*

Proof. Obviously, system X_{015} has the Hamiltonian function $H_1 = a_{05}y^6/6 - b_{40}x^5/5$. Since system X_{015} has the analytic first integral H_1 and its origin O is an isolated singularity, the local neighborhood of O must consist of only finitely many hyperbolic sectors without elliptic and parabolic sectors. In order to determine the number of hyperbolic sectors, we get from the *Bendixson's formula* (see e.g. [7, p.69] and [31, Chapter III, Section 6]) that

$$\mathcal{I}(O) = 1 + \frac{\widehat{e} - \widehat{h}}{2}, \tag{4}$$

where $\mathcal{I}(O)$ is the Poincaré index of the singularity O, $\hat{e}$ is the number of elliptic sectors and $\hat{h}$ the number of hyperbolic sectors adjacent to the singularity O. By Theorem 4.1 of [31, Chapter III], the Poincaré index $\mathcal{I}(O) = 0$ because the sum of degrees of the lowest order terms of two components of the vector field X_{015} is odd. Since $\widehat{e} = 0$, it follows that $\hat{h} = 2$, i.e., there are two hyperbolic sectors adjacent to the singularity O. This proves the lemma. ☐

Lemma 5 shows that the origin of the vector field X_{015} is not a center. Actually, if we only want to prove that the origin of the vector field X_{015} is not a center, the proof can be simplified. It follows from the second equation $y'(t) = b_{40}x^4$ of X_{015} that $y(t)$ is increasing if $b_{40} > 0$ and decreasing if $b_{40} < 0$ for $t \in (-\infty, +\infty)$. Therefore, $y(t)$ is not a periodic function, which yields that X_{015} has no periodic orbits. It is obvious that the origin is not center if $b_{40} = 0$.

We now study the origin of the system X_{021}. Recall that $b_{10} \neq 0$, otherwise X_{021} is essential of degree 3. After the linear transformation of variables and the time rescaling

$$x_1 = \frac{b_{10}}{\sqrt{|b_{10}a_{05}|}}\, x, \quad y_1 = y, \quad dt_1 = \sqrt{|b_{10}a_{05}|}dt,$$

system X_{021} is reduced to its equivalent forms

$$X_{021}^{\pm} : \ \dot{x} = axy^2 \pm y^5, \ \dot{y} = x + by^3,$$

whose origins are nilpotent singularities, where $a = \frac{a_{12}}{\sqrt{|b_{10}a_{05}|}}$, $b = \frac{b_{03}}{\sqrt{|b_{10}a_{05}|}}$ and we write (x_1, y_1) as (x, y) for simplicity.

Lemma 6. *For system $X_{021}^{\pm}$, the following statements hold.*

(*a*) *The origin O of system X_{021}^{+} is not a center.*

(*b*) *System X_{021}^{-} has a center at the origin O if and only if $a = -3b$, $b^2 < \frac{1}{3}$. The vector field X_{021}^{-} has the polynomial first integral*

$$H^{+}(x,y) = \frac{x^2}{2} + bxy^3 + \frac{y^6}{6}. \tag{5}$$

Proof. Suppose that the vector fields, which correspond to $X_{021}^{\pm}$ respectively, are $(P_{\pm}, Q_{\pm})$. It is easy to compute that

$$\frac{\partial P_{\pm}}{\partial x} + \frac{\partial Q_{\pm}}{\partial y} = (a+3b)y^2.$$

By Bendixsons Criteria, system $X_{021}^{\pm}$ has no periodic orbit if $a + 3b \neq 0$. Hence, if $a \neq -3b$, then the origin is not a center.
(*a*) Substituting $x = f_1 := -by^3$ into the first component of X_{021}^{+} gives

$$F_1(y) := (axy^2 + y^5)|_{x=f_1} = (1-ab)y^5.$$

Set

$$G_1(y) := \left(\frac{\partial}{\partial x}(axy^2 + y^5) + \frac{\partial}{\partial y}(by^3)\right)\Big|_{x=f_1} = (a+3b)y^2.$$

If $a = -3b$, i.e., $G_1(y) \equiv 0$, then $F_1(y) = (1+3b^2)y^5$. By [4, Theorem 4.(a)], the origin O of system X_{021}^{+} is not a center provided $a = -3b$.
(*b*) For system X_{021}^{-}, substituting $x = f_1 := -by^3$ into the first component of X_{021}^{-} yields

$$F_2(y) := (axy^2 - y^5)|_{x=f_1} = -(1+ab)y^5.$$

Set

$$G_2(y) := \left(\frac{\partial}{\partial x}(axy^2 - y^5) + \frac{\partial}{\partial y}(by^3)\right)\Big|_{x=f_1} = (a+3b)y^2.$$

When $a = -3b$, we have $G_2(y) \equiv 0$ and $F_2(y) = (-1+3b^2)y^5$. By [4, Theorem 4.(a)], the origin O of system X_{021}^{-} is monodromy if and only if $-1 + 3b^2 < 0$ in the case $a = -3b$. Since X_{021}^{-} has the polynomial first integral (5), it forces that the origin O of system X_{021}^{-} must be a center as $a = -3b$ and $3b^2 < 1$. We complete the proof of statement (*b*), and consequently of the lemma. □

In the following, we consider the invariant curves passing through the origin of system X_{011}.

Lemma 7. *System X_{011} has an invariant curve passing through the origin O.*

Proof. Firstly we use the change

$$x = \tilde{y},\ y = \tilde{x},$$

which transforms the vector field of X_{011} into

$$(6)\quad (\hat{P},\ \hat{Q}) := (b_{04}x^4 + b_{12}yx^2 + b_{20}y^2,\ a_{05}x^5 + a_{13}yx^3 + a_{21}y^2x),$$

and we still write $(\tilde{x}, \tilde{y})$ as (x, y) for simplicity. Using the notation in [21], we have

$$\begin{aligned}(7)\ \eta(x,y) &= x\hat{Q}(x,y) - 2y\hat{P}(x,y)\\ &= a_{05}x^6 + a_{13}x^4y + a_{21}x^2y^2 - 2b_{04}x^4y - 2b_{12}x^2y^2 - 2b_{20}y^3.\end{aligned}$$

Clearly the cubic polynomial

$$\eta(1,\lambda) = a_{05} + (a_{13} - 2b_{04})\lambda + (a_{21} - 2b_{12})\lambda^2 - 2b_{20}\lambda^3,$$

has at least one real zero, denoted by λ_1. Then X_{011} has the invariant curve $x - \lambda_1 y^2 = 0$, see Proposition 4 in [21]. This proves the lemma. □

Proof of Theorem 2. From the analysis before Lemma 5, we know that only the four systems X_{011}, X_{012}, X_{015} and X_{021} possibly have a center at the origin. System X_{012} can be transformed into X_{021} by the change of variables and time scaling

$$(\widetilde{x}, \widetilde{y}, d\widetilde{t}) = (x^2, y, xdt).$$

Lemmas 5 and 7 show that the origins of X_{015} and X_{011} are not centers. Hence, it follows from Lemma 6 that system X_{021} is the unique one which possibly has a center at the origin and that the origin is a center if and only if $a = -3b$ and $b^2 < \frac{1}{3}$.

Finally we prove that the center of system (2) is not isochronous and the period of the periodic orbits is a monotonic function. The first claim follows from the facts that the center of system (2) is nilpotent, and that if a center of polynomial differential system is isochronous, it must be elementary, i.e. the eigenvalues are a pair of pure imaginary numbers, see for instance [4, 20].

Next we prove the monotonicity of the period. From the first integral (5), set

$$\Gamma_h:\quad H^+(x,y) = \frac{x^2}{2} + bxy^3 + \frac{y^6}{6} = h^2,$$

where $h > 0$ and $b^2 < \frac{1}{3}$. Then the periodic orbit Γ_h can be parameterized as

$$(8)\qquad \frac{x + by^3}{\sqrt{2}} = h\cos s,\ \sqrt{\frac{1-3b^2}{6}}\,y^3 = h\sin s,\qquad s \in [0, 2\pi].$$

Using this expression and taking some calculations, we get that the period $T(h)$ of Γ_h is

$$T(h) = \int_{\Gamma_h} \frac{dy}{x+by^3} = \frac{1}{3\sqrt{2}}\left(\frac{6}{1-3b^2}\right)^{\frac{1}{6}} h^{-\frac{2}{3}} \int_0^{2\pi} (\sin s)^{-\frac{2}{3}} ds, \tag{9}$$

with

$$\int_0^{2\pi} (\sin s)^{-\frac{2}{3}} ds = \frac{2\sqrt{\pi}\text{Gamma}(\frac{1}{6})}{\text{Gamma}(\frac{2}{3})}.$$

Clearly the period $\Gamma(h)$ of the periodic orbits inside the period annulus of the center O is monotonic in h. We complete the proof of the theorem. □

4. Topological phase portraits of quintic quasi–homogeneous vector fields having a center

For the quasi–homogeneous quintic coprime polynomial system (2) with a center at the origin, we have the first integral (5). But they are not enough to get the global phase portrait of system (2). Since the origin is the unique finite singularity of system (2) and it is a center, so we need to use the Poincaré compactification to study the dynamics of system (2) at the infinity. For more information on the Poincaré compactification, see e.g. [8].

Taking respectively the Poincaré transformations $x = 1/z$, $y = u/z$ and $x = v/z$, $y = 1/z$ together with the time variables $d\tau = dt/z^4$, system (2) around the equator of the Poincaré sphere can be written respectively in

$$\begin{aligned} \frac{du}{d\tau} &= u^6 + (b-a)u^3z^2 + z^4 := P_1(u,z), \\ \frac{dz}{d\tau} &= u^2z(u^3 - az^2) := Q_1(u,z), \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{dv}{d\tau} &= -1 + avz^2 - bvz^2 - v^2z^4, \\ \frac{dz}{d\tau} &= -z^3(b + vz^2). \end{aligned} \tag{11}$$

System (10) has the unique singularity $E_1 = (0,0)$ on the u-axis, which is located at the end of the x–axis and is a singularity of system (2) at infinity. System (11) has no singularities at the end of the y–axis. So system (2) has a unique singularity at the infinity.

For discussing the singularity at the infinity, we need to introduce the methods of generalized normal sectors [26]. Consider the analytic differential system

$$\begin{aligned} \dot{x} &= X_m(x,y) + \Phi_m(x,y) := X(x,y), \\ \dot{y} &= Y_m(x,y) + \Psi_m(x,y) := Y(x,y), \end{aligned} \tag{12}$$

where $X_m(x,y)$ and $Y_m(x,y)$ are homogeneous polynomials of degree $m \geq 1$ and do not simultaneously and identically vanish, and $\Phi_m(x,y)$, $\Psi_m(x,y) = o(r^m)$ as $r = \sqrt{x^2+y^2} \to 0$. Let O be an isolated singularity of (12). In order to see whether O is monodromy, by Lemmas 1 and 3 in [24, Chapter 2] we only need to discuss the orbits along exceptional directions of system (12) at O. Applying the polar coordinate changes $x = r\cos\theta$ and $y = r\sin\theta$, system (12) can be written in

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H_m(\theta)+o(1)}{G_m(\theta)+o(1)}, \quad \text{as } r \to 0, \tag{13}$$

where

$$G_m(\theta) = \cos\theta Y_m(\cos\theta, \sin\theta) - \sin\theta X_m(\cos\theta, \sin\theta). \tag{14}$$

Hence a necessary condition for $\theta = \theta_0$ to be an exceptional direction is $G_m(\theta_0) = 0$.

The next lemma is obtained in [26, Lemmas 4].

Lemma 8. *Let $\Delta\widehat{AOB}$ be an open quasi–sector in the first (resp. second) quadrant, which is limited by two smooth simple curves $\widehat{OA}$, $\widehat{OB}$ and a circular arc $\widehat{AB}$ centered at O. If $Y/X < 0$ (resp. > 0) in $\Delta\widehat{AOB}$, system (12) has no orbits connecting the origin O in this region.*

Lemma 9. *Under the conditions $a = -3b$ and $b^2 < \frac{1}{3}$, system (10) has a unique orbit leaving from E_1 and also a unique orbit approaching E_1 in the positive time.*

Proof. The vector field (10) is symmetric with respect to the u–axis, because $P_1(u,-z) = P_1(u,z)$ and $Q_1(u,-z) = -Q_1(u,z)$. Thus, we only need to discuss the orbits connecting the origin E_1 of (10) in the first and the second quadrants. The invertible transformation

$$u_1 = u, \ z_1 = z^2$$

for $z \geq 0$, reduces system (10) to

$$\begin{aligned} \dot{u} &= u^6 + 4bu^3z + z^2 := P_2(u,z), \\ \dot{z} &= 2u^2z(u^3+3bz) := Q_2(u,z), \end{aligned} \tag{15}$$

where we still write (u_1, z_1) as (u,z) for simplicity. The origin E_1 of (10) corresponds to the origin $E_2 = (0,0)$ of (15). From (14), we get $G_2(\theta) = -\sin^3\theta$ for (15), which has exactly two roots 0 and π. Except these two directions, there are no directions along which system (15) has orbits connecting E_2. We will construct some related open quasi–sectors to determine how many orbits of (15) connecting E_2 in the first and the second quadrants.

Observing that system (15) has three horizontal isoclines: the u-axis, the z-axis and

$$\mathcal{H}_2 := \left\{(u,z) \in \mathbb{R}^2 : u = (-3b)^{\frac{1}{3}} z^{\frac{1}{3}}, \ 0 < r < \ell\right\},$$

for $b \neq 0$, where $\ell > 0$ is a sufficiently small constant. Set

$$\mathcal{U}^+ := \{(u,z) \in \mathbb{R}^2 : z = 0,\ u > 0,\ 0 < r < \ell\},$$
$$\mathcal{U}^- := \{(u,z) \in \mathbb{R}^2 : z = 0,\ u < 0,\ 0 < r < \ell\}.$$

The possible vertical isoclines are

$$\mathcal{V}_2^\pm := \left\{(u,z) \in \mathbb{R}^2 : u = v_2^\pm z^{\frac{1}{3}},\ 0 < r < \ell\right\},$$

where $v_2^\pm = (-2b \pm \sqrt{4b^2-1})^{\frac{1}{3}}$ for $b^2 > \frac{1}{4}$. Obviously, the isoclines $\mathcal{H}_2$ and $\mathcal{V}_2^\pm$ are tangent to the u–axis at the origin. Set

$$\mathcal{L}^\pm := \left\{(u,z) \in \mathbb{R}^2 : z = \pm\, \sigma u,\ 0 < r < \ell\right\},$$

where $\sigma > 0$ is a small constant. Hence, if there exist orbits of system (15) connecting E_2 along the direction of the u–axis in the first and the second quadrants, then near the origin the orbits must lie in the sector regions $\Delta\widehat{\mathcal{L}^+ E_2 \mathcal{U}^+}$ and $\Delta\widehat{\mathcal{L}^- E_2 \mathcal{U}^-}$. Next we determine the positions of the isoclines $\mathcal{H}_2$ and $\mathcal{V}_2^\pm$ via $v_2^\pm$ as a function of b, and use it to construct a quasi–sector region with whose boundaries formed by orbits, line segments $\mathcal{L}^\pm$ or isoclines $\mathcal{H}_2$ and $\mathcal{V}_2^\pm$. We distinguish four cases: $b < -\frac{1}{2}$; $b > \frac{1}{2}$; $|b| = \frac{1}{2}$ and $|b| < \frac{1}{2}$. Recall that we are in the conditions $a = -3b$ and $b^2 < \frac{1}{3}$.

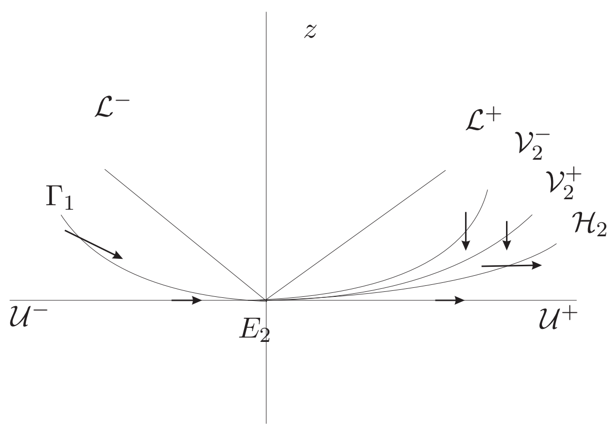

FIGURE 2. Directions of vector field for system (15) as $b < -\frac{1}{2}$.

Now, we discuss the case $b < -\frac{1}{2}$, which is most complicated among the four cases. When $a = -3b$ and $b^2 < \frac{1}{3}$, we have $-\frac{1}{\sqrt{3}} < b < -\frac{1}{2}$. Moreover, $-3b > -2b \pm \sqrt{4b^2-1}$ is equivalent to $b^2 < \frac{1}{3}$. This implies that all isoclines lie in the first quadrant only, and $v_2^- < v_2^+ < (-3b)^{\frac{1}{3}}$. This last condition is equivalent to say that the horizontal isocline $\mathcal{H}_2$ lies on the right hand of the vertical isoclines $\mathcal{V}_2^\pm$ near the origin, see Figure 2. We can check that $\dot z < 0$

on $\mathcal{V}_2^{\pm}$ and $\dot{u} > 0$ on $\mathcal{H}_2$. It follows that $\dot{u} > 0$ and $\dot{z} > 0$ in $\Delta\widehat{\mathcal{U}^+E_2\mathcal{H}_2}$; $\dot{u} > 0$ and $\dot{z} < 0$ in $\Delta\widehat{\mathcal{V}_2^+E_2\mathcal{H}_2}$; $\dot{u} < 0$ and $\dot{z} < 0$ in $\Delta\widehat{\mathcal{V}_2^+E_2\mathcal{V}_2^-}$; $\dot{u} > 0$ and $\dot{z} < 0$ in $\Delta\widehat{\mathcal{V}_2^-E_2\mathcal{L}^+}$; and $\dot{u} > 0$ and $\dot{z} < 0$ in $\Delta\widehat{\mathcal{U}^-E_2\mathcal{L}^-}$.

Lemma 8 guarantees that no orbits connect E_2 in $\Delta\widehat{\mathcal{V}_2^+E_2\mathcal{H}_2}$ and $\Delta\widehat{\mathcal{V}_2^-E_2\mathcal{L}^+}$, respectively. We claim that there are also no orbits connecting E_2 in each of $\Delta\widehat{\mathcal{U}^+E_2\mathcal{H}_2}$, $\Delta\widehat{\mathcal{V}_2^+E_2\mathcal{V}_2^-}$ and $\Delta\widehat{\mathcal{U}^-E_2\mathcal{L}^-}$. Then there are no orbits connecting E_2 in the first and second quadrants. By contrary, if there exists an orbit Γ_1 in the first or second quadrant connecting E_2 along the u–axis, then the tangent vector to Γ_1 is not the vector associated to system (15), a contradiction.

We now prove this fact. We first consider the case that Γ_1 lies in $\Delta\widehat{\mathcal{U}^-E_2\mathcal{L}^-}$. Since Γ_1 is tangent to the u–axis at E_2, it can be expressed in the form

$$z = \gamma_1(u) := \beta_1 u^{k_1} + o(|u^{k_1}|), \tag{16}$$

where $k_1 > 1$, and either $\beta_1 > 0$ if $u^{k_1} > 0$, or $\beta_1 < 0$ if $u^{k_1} < 0$. In the following, we prove that

$$\frac{\dot{z}}{\dot{u}}\Big|_{\gamma_1} > \gamma_1'(u). \tag{17}$$

Notice that $\dot{u} > 0$ and $\dot{z} < 0$ in $\Delta\widehat{\mathcal{U}^-E_2\mathcal{L}^-}$. If $k_1 > 3$, the inequality (17) is equivalent to

$$-\beta_1 k_1 u^{k_1+5} > -2\beta_1 u^{k_1+5} + o(|u^{k_1+5}|),$$

which is obviously true. If $k_1 = 3$, the inequality (17) is reduced to

$$3\beta_1^2 + 6b\beta_1 + 1 + O(|u|) > 0,$$

which is also true because $b^2 < \frac{1}{3}$. If $1 < k_1 < 3$, the inequality (17) is simplified as

$$-\beta_1^3 k_1 u^{3k_1-1} > 2a\beta_1^2 u^{2k_1+2} + o(|u^{2k_1+2}|),$$

which is true because $3k_1 - 1 < \min\{2k_1+2, 5+k_1\}$. Hence, the inequality (17) is proved. Using the same arguments we can obtain (17) when Γ_1 lies in $\Delta\widehat{\mathcal{V}_2^+E_2\mathcal{V}_2^-}$, and $\frac{\dot{z}}{\dot{u}}|_{\gamma_1} < \gamma_1'(u)$ when Γ_1 lies in $\Delta\widehat{\mathcal{U}^+E_2\mathcal{H}_2}$. Hence the claim is proved.

We have proved that in the case $b < -\frac{1}{2}$, system (15) has no orbits which connect E_2 and are located in the first and the second quadrants. The other three cases can be proved by the same arguments as $b < -\frac{1}{2}$, and the details are omitted.

Summarizing the above proofs, we achieve that system (15) has exactly two orbits connecting E_2, one leaving from E_2 and the other approaching E_2 in the positive time. Since the infinity is invariant, the two orbits must coincide with respectively the positive and negative half parts of the u–axis. We complete the proof of the lemma. $\square$

Proof of Theorem 3. Since system (2) has a unique finite singularity at the origin which is a center, and a unique singularity at infinity, by Lemma 9 we get that the origin of system (2) is a global center. So the phase portrait of system (2) is that given in Figure 1. We complete the proof of the theorem. □

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