Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 13: Curl and divergence in two dimensions

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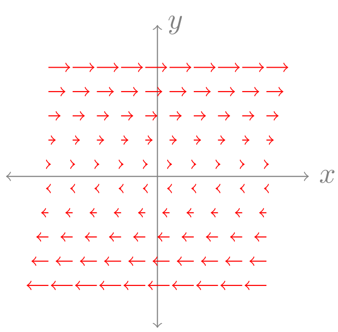
1 Curl, or circulation density

For the purposes of intuition, we imagine that our vector field $\mathbf{F}$ represents the current on the surface of some body of water. If you toss a small chip of wood into the water, it will float on the surface. What else will it do?

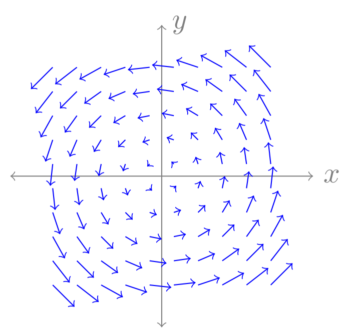
If the water is perfectly still, it will do nothing. If there is a current in the water, we can say two things about its movement:

- First, the chip of wood will move with the current. If the body of water is a river, the chip of wood will float downstream. If the body of water is a sink full of water, the chip of wood will circle the drain. If we imagine the water doing other, less plausible things, the chip of wood might have other global behavior.
- There is another effect we can see, which is due to the fact that the chip of wood is not a perfect point mass; it's small, but it has some nonzero length. If the water has a different velocity at different points on the surface, then it's possible that different parts of the chip of wood will get pushed by the water at different speeds. This will cause the chip of wood to turn in place as it floats.

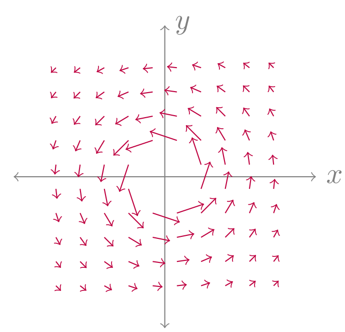
It is the second effect that we will be interested in quantifying today. Let's begin by looking at some examples.



(a) $\mathbf{F} = y\,\mathbf{i}$



(b) $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$



(c) $\mathbf{F} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$

Figure 1: Three vector fields as examples of circulation density

First, consider the vector field $\mathbf{F}_1 = y\,\mathbf{i}$ shown in Figure 1a. This kind of vector field is sometimes described as a "shearing flow". Here, it's easy to imagine a chip of wood turning in place somewhere near the x-axis: the top of it gets pushed right, and the bottom gets pushed left, so it spins. Actually,

[1]This document comes from the Math 3204 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php

the same effect happens at every other point, too; if you're above the x-axis, for instance, the top of an object gets pushed faster than the bottom, so it spins.

Next, let's compare the two vector fields Figure 1b and Figure 1c. These both have a global rotational effect: the chip of wood would be circling the whirlpool. But in Figure 1b, since the speed increases as we move away from the origin, the chip of wood would also start spinning. Figure 1c, which shows the $\mathrm{d}\theta$ example from the previous lecture, does not do this; the chip of wood would keep its orientation fixed while going around the whirlpool.

We can quantify this effect with a parameter of the vector field that we can call **curl**[2] or **circulation density**. This is a number that, at every point (x, y), determines how much spin $\mathbf{F}$ puts on an object at (x, y). To figure out what this number should be, we imagine taking a circulation integral that goes around a point in a tiny counterclockwise loop. Of course, the result will depend on the size of the loop somehow—but in what way?

Let's start with a simple case: $\mathbf{F}$ will be a vector field with a linear equation

$$\mathbf{F} = \mathbf{a} + \mathbf{b}x + \mathbf{c}y = (a_1 + b_1 x + c_1 y)\,\mathbf{i} + (a_2 + b_2 x + c_2 y)\,\mathbf{j}.$$

To understand the curl of this vector field $\mathbf{F}$ at a point (x, y), we will take an integral that goes in a square around the point, with corners at $(x \pm h, y \pm h)$, as in Figure 2.

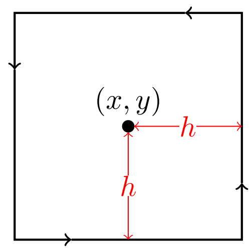


Figure 2: A square circulation integral

The right side of the square is parameterized by $\mathbf{r}(t) = (x + h, y + t)$ where $t \in [-h, h]$. We have $\mathbf{F}(\mathbf{r}(t)) = \mathbf{a} + (x + h)\mathbf{b} + (y + t)\mathbf{c}$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = a_2 + (x + h)b_2 + (y + t)c_2$. We take the integral and get

$$\int_{t=-h}^{h} (a_2 + (x + h)b_2 + (y + t)c_2)\,\mathrm{d}t = 2h\Big(a_2 + xb_2 + y_0 c_2\Big) + 2h^2 b_2.$$

There is a lot going on here, but things will become simpler when we compare this to the left side of the square. Here, we have $\mathbf{r}(t) = (x - h, y - t)$ where $t \in [-h, h]$, so $\mathbf{F}(\mathbf{r}(t))$ is now $\mathbf{a} + (x - h)\mathbf{b} + (y - t)\mathbf{c}$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is now $-\mathbf{j}$. As a result, this integral becomes

$$\int_{t=-h}^{h} -(a_2 + (x - h)b_2 + (y - t)c_2)\,\mathrm{d}t = -2h\Big(a_2 + xb_2 + yc_2\Big) + 2h^2 b_2.$$

Almost everything in these two integrals cancels, and we are just left with $4h^2 b_2$ as the net contribution!

[2] Actually, "curl" is also referred to a different quantity, which is a vector parameter of a vector field in $\mathbb{R}^3$. To disambiguate, some sources refer to the scalar quantity we discuss today as "the $\mathbf{k}$-component of curl"—but other sources just refer to both as "curl".

Similarly, most of the integral along the top side will cancel with most of the integral along the bottom side. I will skip the details, since they're very similar, but we'll get

$$\int_{t=-h}^{h} -(a_1 + (x+t)b_1 + (y+h)b_1)\,\mathrm{d}t + \int_{t=-h}^{h} (a_1 + (x+t)b_1 + (y-h)c_1)\,\mathrm{d}t$$

which simplifies to $-4h^2c_1$ as the net contribution. Putting the two pieces together, we get $(b_2 - c_1)4h^2$.

The quantity $4h^2$ is the area of the square. So we can think of the quantity $b_2 - c_1$ as the circulation *density* of the vector field $\mathbf{F}$: it's a quantity, that when you multiply it by an area, gives a circulation.

From here, we can generalize to an arbitrary vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. We would not be able to get an equally nice *exact* formula for the circulation of $\mathbf{F}$ around the square. But if we take the first-order Taylor series approximation of $\mathbf{F}$ near a point (x, y), we will get a vector field with a linear equation like the one we dealt with—with $b_1 = \frac{\partial M}{\partial x}$, $c_1 = \frac{\partial M}{\partial y}$, $b_2 = \frac{\partial N}{\partial x}$, and $c_2 = \frac{\partial N}{\partial y}$. For the linear approximation, we will get a circulation of $(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})4h^2$. There will be an error in the approximation, but the error term on $\mathbf{F}$ will be quadratic in h, and integrated over a curve of length $8h$, for a result that's *cubic* in h. As a result,

$$\lim_{h\to 0} \frac{\int_{\square} \mathbf{F}\cdot \mathrm{d}\mathbf{r}}{4h^2} = \lim_{h\to 0} \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)4h^2 + O(h^3)}{4h^2} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

(I'm using $O(h^3)$ to stand in for an error term that is cubic in h, and $\square$ to stand in for the curve that goes around the square in Figure 2.)

So it makes sense to say that in general, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ is the circulation density of $\mathbf{F}$ at a point.

This also has connections to what we've previously done:

- By the component test, if $\mathbf{F}$ is conservative, then $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$. This makes sense. For conservative fields, all circulation integrals around closed curves will be 0, and that includes circulation integrals around our tiny squares.

  The quantity $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ could be reasonably interpreted as a measure of how much $\mathbf{F}$ fails to be conservative.

- We have seen the difference $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ once before. When we were working out a general formula for the exterior derivative of a 1-form $M\,\mathrm{d}x + N\,\mathrm{d}y$, we got

  $$\mathrm{d}(M\,\mathrm{d}x + N\,\mathrm{d}y) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y.$$

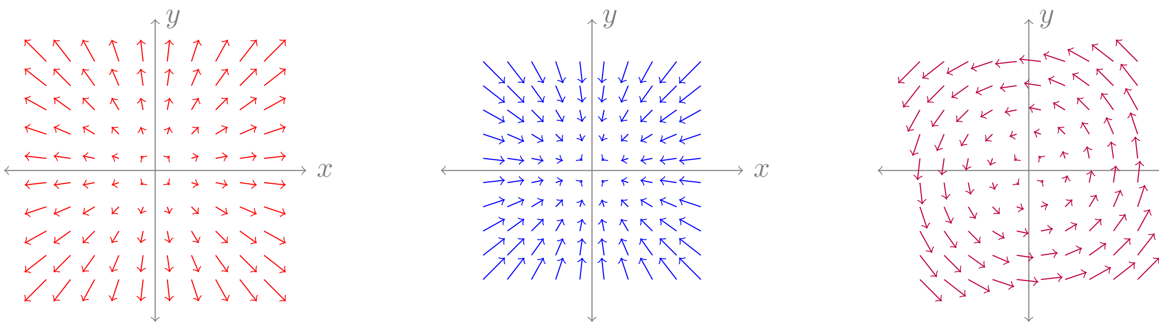
  The exterior derivative gave us the circulation density!

For now, we will write this quantity $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ as $\operatorname{curl}(\mathbf{F})$ and call it the curl of $\mathbf{F}$ or the circulation density of $\mathbf{F}$. A word of warning: right now, we are only considering vector fields in $\mathbb{R}^2$, and talking about curl will become more complicated when we get to $\mathbb{R}^3$.

2 Divergence, or flux density

We can think about flux integrals in a similar way, except that currents of water flow are a somewhat weird metaphor to use. Suppose you have a closed curve C on the surface of a body of water. What would it mean for the current to have a positive outward flux across C? It would mean that, on net, water is leaving the region bounded by C. This shouldn't generally be happening over any length of time.

Air currents (like a wind map) are a different matter. If the region bounded by C is a high-pressure area that's decreasing in pressure, then we *do* expect an outward flux across C: for the air pressure to decrease, air must flow out, allowing the remaining air inside C to be less dense.



(a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\mathbf{F} = -x\,\mathbf{i} - y\,\mathbf{j}$ (c) $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure 3: Three vector fields as examples of flux density

This is exactly the sort of thing that's happening, for example, in Figure 3a. If the air is expanding by a constant amount at *every* point of the plane, then we'll get a vector field pointing outward from some central fixed point. The larger the region where this is happening, the higher the magnitude of the vector field around the boundary will be (because that means more air is trying to get out). This vector field is a lot like what we'd see in an explosion.

Figure 3b is the reverse of this. Here, we would get a negative outward flux across every closed curve.

Figure 3c is a vector field we already looked at once today. Here, though the vector field is going faster and faster around the origin as we go out, there's no expansion or contraction going on. If we took a flux integral across any closed curve, we would get 0.

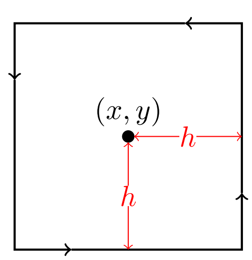


Figure 4: A square flux integral

Just as before, we can measure this by a quantity at every point, which we will call **divergence** or **flux density** and denote div $\mathbf{F}$. We can try to understand what this should be, as before, by looking at the outward flux across the boundary of a very small square. What will we get?

Let's try to derive a formula for div $\mathbf{F}$ by a different, less rigorous and more intuitive argument. Suppose that, once again, we draw a curve that goes in a square with corners $(x \pm h, y \pm h)$ around a point (x, y). (Figure 4 shows this square, though it's the same picture as Figure 2 was.)

The right side of the square goes from $(x+h, y-h)$ to $(x+h, y+h)$, with normal vector $\mathbf{i}$. Though $\mathbf{F}$ varies along this segment, on average it will be equal to $\mathbf{F}(x+h, y)$: its value at the midpoint of the segment. So we can estimate the flux across the right side of the square as $\mathbf{F}(x+h, y) \cdot \mathbf{i} = M(x+h, y)$. Actually, this is $M(x+h, y) \cdot 2h$, because this flux is crossing a segment of length $2h$.

For the left side of the square, the outward normal vector is $-\mathbf{i}$, and the value of $\mathbf{F}$ is on average $\mathbf{F}(x-h, y)$, and we will get a flux of $-M(x-h, y) \cdot 2h$.

The net contribution from these two sides is $\Big(M(x+h, y) - M(x-h, y)\Big) \cdot 2h$, or

$$\frac{M(x+h, y) - M(x-h, y)}{2h} \cdot 4h^2 \approx \frac{\partial M}{\partial x} \cdot 4h^2.$$

Similarly, from the top and bottom side, we will get a net outward flux of $\frac{\partial N}{\partial y} \cdot 4h^2$. Factoring out the area of the square, $4h^2$, we see that it makes sense to define the flux density as $\operatorname{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

We can check by calculation what $\operatorname{div} F$ will be in the three vector fields of Figure 3: we get $\operatorname{div} F = 2$ at every point in Figure 3a, $\operatorname{div} F = -2$ at every point in Figure 3b, and $\operatorname{div} F = 0$ at every point in Figure 3c.