Math 3204: Calculus IV[1]	Mikhail Lavrov

Lecture 7: Vector fields

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1 Introduction to vector fields

1.1 Definitions and examples

Mathematically, a **vector field** (in $\mathbb{R}^2$, or $\mathbb{R}^3$, or $\mathbb{R}^n$) is a function that assigns a vector to every point. In two dimensions, this is a function $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ which assigns a vector $\mathbf{F}(x, y) = (M(x, y), N(x, y))$ to every point (x, y). In three dimensions, this is a function $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ which assigns a vector $\mathbf{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$ to every point (x, y, z).

Here's an example of a vector field you may have already seen: wind maps. These are radar maps which indicate the way the wind is blowing (in terms of direction and magnitude) in a region. If you look at wind maps on your computer[2] you get to see animations that make it very clear how the air is moving.

On paper, the best we can do is to draw arrows, as in Figure 1, which you can imagine to be the wind map of the eye of a hurricane centered at $(0, 0)$. We cannot map the behavior of the wind everywhere, so we map it at many evenly-spaced points. By convention, the arrow *starting* at a point (x, y) points in the direction that the wind is blowing at (x, y), and the length of the arrow is proportional to the wind speed.

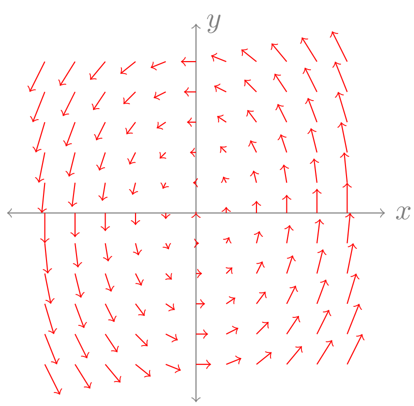


Figure 1: Visualizing a vector field with arrows at evenly spaced points.

More generally, we can define **velocity fields** of particles like air or water: this is a vector field $\mathbf{F}$ where the value $\mathbf{F}(x, y)$ tells us the velocity of particles at the point (x, y). We can do this in three dimensions as well! In principle, our wind maps should be three-dimensional vector fields where

[1] This document comes from the Math 3204 course webpage: `http://facultyweb.kennesaw.edu/mlavrov/courses/3204-fall-2023.php`

[2] Here is an example: `https://www.accuweather.com/en/us/national/wind-flow`. Of course, I am not affiliated with AccuWeather and make no promises about the future of this URL.

$\mathbf{F}(x, y, z)$ tells us the three-dimensional velocity of wind at a point (x, y, z). We don't do this in practice because this would be very hard to visualize, and because the (x, y)-component of wind is usually more significant.

Another important example from physics is **force fields**. These are not glowing barriers that stop lasers. Force fields are vector fields that indicate the direction and magnitude of physical forces (like gravity or magnetism) at a point.

1.2 Intuition and notation

Not every function $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ makes sense as a vector field. We really want the output of $\mathbf{F}$ to be a vector that points in a direction relative to the point we give as an input. To give a non-example: if we assign each point on a two-dimensional map the pair (temperature, pressure), that's a function $\mathbb{R}^2 \to \mathbb{R}^2$, but it does not have a good vector field interpretation.

The distinction we're drawing is that a point represents a *position*, and a vector represents a *difference* between two positions. Many operations that make sense for vectors do not make sense for points:

- The $\mathbf{0}$ vector is always a very important vector. For example, on a wind map, it represents a complete absence of wind. On the other hand, the point $(0, 0, 0)$ is just an arbitrary point chosen so that we can put coordinates on $\mathbb{R}^3$; there is nothing special about it.
- We can add, subtract, and scale vectors; for example, we can ask, "What if the wind were blowing in the same direction, but twice as much?" It does not make sense to talk about multiplying a point by two, except indirectly (in relation to the origin).

In this class, I will distinguish points and vectors by the notation I use. (I will not require you to do this; this is just an aid to help you keep track of the quantities we're dealing with.) I will write:

- (x, y, z) for a point with coordinates x, y, z.
- $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ for the vector pointing from $(0, 0, 0)$ to (x, y, z).

In particular, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard basis vectors of $\mathbb{R}^3$: they are the vectors pointing from $(0, 0, 0)$ to $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively.

I have already done this once: when dealing with a parameterization $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$, the value of $\mathbf{r}(t)$ is a point $(x(t), y(t), z(t))$, but its derivative $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is a vector $\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathbf{k}$.

Similarly, when dealing with a vector field $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$, I will not write

$$\mathbf{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z)).$$

Instead, I will write

$$\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}.$$

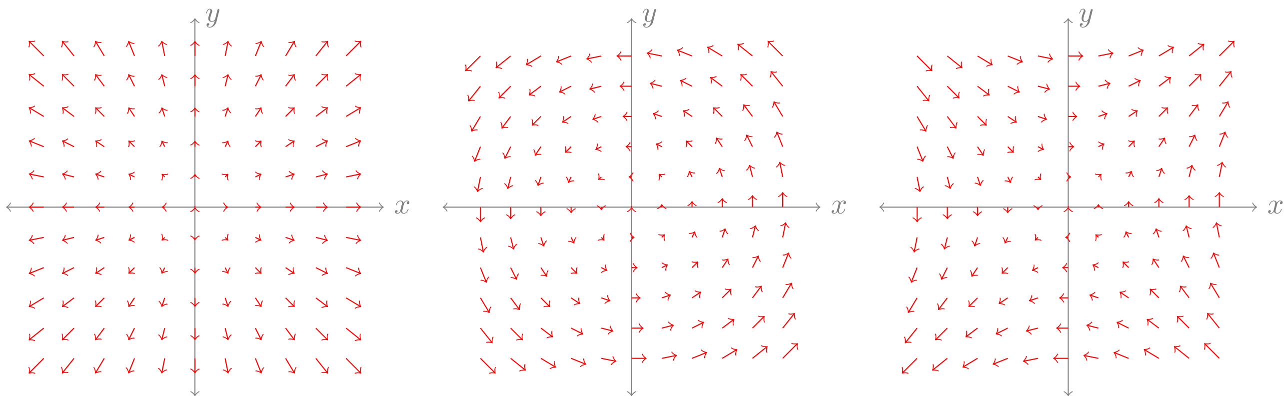
(You also see me dropping the arguments of the functions, writing $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ rather than $\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}$ and $M\,\mathbf{i}$ rather than $M(x, y, z)\,\mathbf{i}$. This is just to simplify notation; these things are still functions of the variables they should be functions of.)

2 Doing some math with vector fields

2.1 A few examples

Figure 2 shows three vector fields in $\mathbb{R}^2$ with simple formulas. We will often use these as examples, and it helps to see them and where their properties come from.

(You should make sure that you're able to draw at least a sketch of these vector fields in examples like these. Of course, if the vector fields get too complicated, we don't want to draw them by hand.)



(a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ (c) $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$

Figure 2: Three simple vector fields in $\mathbb{R}^2$.

Figure 2a shows the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$. All the vectors here point directly away from the origin, and their magnitude is proportional to the distance from the origin. (This is a good time to point out that we draw vector fields with the arrows *proportional* to the magnitude of the vector, but usually not *equal* in magnitude to that vector. If we did that, our arrows would be much longer; they'd overlap, and the diagram would be impossible to read!

Figure 2b shows the vector field $\mathbf{F} = -y\,\mathbf{i} + x\mathbf{j}$. First of all, this vector field clearly has some amount of counterclockwise rotation happening (which we will eventually learn how to quantify). As we go further out from the origin, the "angular velocity" of the vectors remains the same, which means that the magnitude in the usual sense eventually becomes very big.

Figure 2c has mixed behavior. In the first and third quadrant, the vector field points away from the origin; in the second and fourth quadrant, the vector field points toward the origin.

2.2 The gradient field

A scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is often called a **scalar field**: just as a vector field gives us a vector at every point of $\mathbb{R}^3$, a scalar field gives us a scalar (that is, a real number) at every point of $\mathbb{R}^3$.

The **gradient field** (or just **gradient**) $\boldsymbol{\nabla} f$ of a scalar function f is a a vector field:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

By taking each component to be the rate of change of the corresponding variable, we end up getting a vector that points in the direction where f is increasing the most.

Here's a two-dimensional practical example. Suppose that f is a topographical map of a region: $f(x, y)$ gives us the height above sea level at coordinates (x, y). (For this example, we ignore the global curvature of the Earth.) Then at each point (x, y), $\boldsymbol{\nabla} f$ is a vector pointing in the direction of the steepest incline; its magnitude $\|\boldsymbol{\nabla} f\|$ is the tangent slope in that direction. When does this gradient field play a role? Well, for example, after heavy rainfall: when it rains, the water will flow along the steepest downward slope, so its movement will be described by the vector field $-\boldsymbol{\nabla} f$.

In Figure 2, two of the diagrams show gradient fields. Figure 2a is the gradient field of $f(x, y) = \frac{1}{2}(x^2 + y^2)$. (In the topographical analogy, f represents a parabolic hole in the ground, centered at the origin. The direction of steepest ascent is away from the origin, and the further you go, the steeper the climb is.) Figure 2c is the gradient field of $f(x, y) = xy$. (In the topographical example, this is a saddle point—mountain passes often have approximately this shape. To the northeast and southwest are two mountains; to the northwest and southeast, we can descend into a valley.)

Not every vector field is a gradient field. It is not yet obvious why, but the vector field in Figure 2b is not the gradient field of any function f. (We will see how to determine this later!)

2.3 Work

Work is an unfortunately frustrating concept from physics. We will discuss it now because it motivates the vector line integrals that we will see in the coming lectures.

In this setting, our vector field $\mathbf{F}$ should be a force field; the easiest force to imagine as an example is gravity. We are tracking the path of some object which our force is capable of pushing around somehow.

However, the object does not necessarily move the way that the force pushes it! Some examples:

- Suppose an object is sliding down a slope. Gravity is pulling it down; however, the object also ends up moving horizontally, not in the direction of the force we're looking at.
- Suppose you throw a ball high up into the air. Initially, the ball is moving upward: directly opposite to the effect of gravity!

The work done by a force measures how much the force is "helping" accomplish the actual motion we observe. Here's how this plays out in our two examples:

- Whether an object slides down a steep slope or a shallow slope, if its altitude decreases by 1 food, then gravity "contributed" the same amount of work to the result.
- As a ball flies up, gravity is doing negative work: the opposite of helping. As the ball goes down, gravity is doing positive work. If the ball starts and ends in your hand, the net amount of work done by gravity is 0.

You may object that intuitively, gravity is doing a lot more than 0 in the service of making sure that the ball ends up back in your hand! That is good intuition for *something*, but that something is not what the physicists are calling work.

You shouldn't be asking yourself: how did the ball travel, and how would it have traveled in a world without gravity? That is an unrelated hypothetical. Instead, imagine that the path of the ball is fixed ahead of time, and ask: how much easier or harder is it to accomplish that in a world with gravity, as opposed to in a world without gravity? The answer is: the first segment of the ball's path is harder to accomplish when gravity exists (so gravity is doing negative work). The second segment is easier to accomplish when gravity exists (so gravity is doing positive work). Overall, the path is equally hard to follow with or without gravity.

In the very simplest case, suppose our vector field is constant: a fixed force $\mathbf{F}$ no matter where we are in space.

- If the object moves by a vector $\mathbf{s}$ parallel to $\mathbf{F}$ (and in the same direction), then the work done is the product $\|\mathbf{F}\|\,\|\mathbf{s}\|$: how much was the force helping at each step, multiplied by how far the object had to go.
- If the object moves by a vector $\mathbf{s}$ perpendicular to $\mathbf{F}$, then the work done is 0: motion perpendicular to $\mathbf{F}$ is equally hard with or without the presence of $\mathbf{F}$.

  Note that $\mathbf{s}$ is perpendicular to $\mathbf{F}$ if and only if the dot product $\mathbf{F}\cdot\mathbf{s}$ is 0.
- If the vector $\mathbf{s}$ makes an angle of θ with $\mathbf{F}$, that corresponds to a mix of the previous cases: motion parallel to $\mathbf{F}$ over some length s_1, and motion perpendicular to $\mathbf{F}$ over some length s_2. Ignoring the second component, we get $\|\mathbf{F}\|\,s_1$ for the amount of work done.

How do we figure out what s_1 and s_2 are? One way to do it is with trigonometry: $s_1 = \|s\|\cos\theta$ and $s_2 = \|s\|\sin\theta$. This gives us $\|F\|\,\|s\|\cos\theta$ for the amount of work done.

Algebraically, this is equal to the dot product $\mathbf{F}\cdot\mathbf{s}$. This is much more convenient in practice, since if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ and $\mathbf{s} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\mathbf{F}\cdot\mathbf{s}$ is the sum of products $Mx + Ny + Pz$. This is the definition we will almost always want.

Here's a proof that the dot product really does compute the same quantity. The parallel-perpendicular decomposition of $\mathbf{s}$ is

$$\mathbf{s} = \underbrace{\frac{\mathbf{F}\cdot\mathbf{s}}{\|\mathbf{F}\|^2}\mathbf{F}}_{\text{parallel}} + \underbrace{\left(\mathbf{s} - \frac{\mathbf{F}\cdot\mathbf{s}}{\|\mathbf{F}\|^2}\mathbf{F}\right)}_{\text{perpendicular}}.$$

To check this, check that the dot product of $\mathbf{F}$ with the perpendicular component really is 0.

Therefore $s_2 = \|s\|\cos\theta$ is just the length of the parallel component $\frac{\mathbf{F}\cdot\mathbf{s}}{\|\mathbf{F}\|^2}\mathbf{F}$, which simplifies to $\frac{\mathbf{F}\cdot\mathbf{s}}{\|F\|}$. Multiplying by $\|\mathbf{F}\|$ gives us $\mathbf{F}\cdot\mathbf{s}$ as the answer.