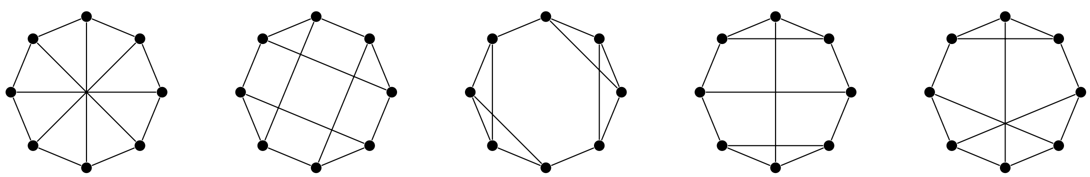
Graph Theory Homework 4

Mikhail Lavrov

due Friday, October 8, 2021

1 Short answer

1. When we were discussing regular graphs in class, we found the following five connected 3-regular graphs on 8 vertices.



   There is a short argument for why none of these graphs have cut vertices (which applies to all of them at once). What is it?

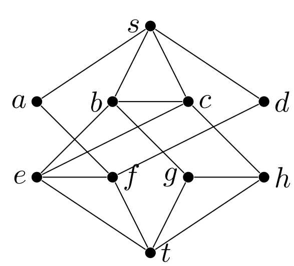
2. The *friendship graph* F_n has $2n + 1$ vertices x, $y_1, \dots, y_n$, and $z_1, \dots, z_n$. The vertex x is adjacent to all other vertices; also, vertices y_i and z_i are adjacent for $i = 1, \dots, n$. There are no other edges.

   (a) Draw a diagram of the friendship graph F_4.

   (b) What are the blocks of the friendship graph F_4? Label them in your diagram.

   (c) How many blocks does the friendship graph F_n have in general, in terms of n?

3. Let G be the graph below.



   (a) Find a 3-vertex $s - t$ cut in G.

   (b) Find three internally disjoint $s - t$ paths in G. Give a one-line explanation of why the existence of these paths means that $\kappa(s, t)$ is at least 3.

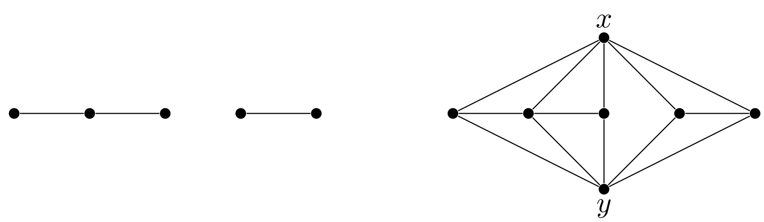
2 Proof

4. Let G be a connected graph with n vertices and n edges. Prove that G has exactly one cycle. (That is, exactly one subgraph which is a cycle graph.)

   You have already written a rough draft of this solution; now, write a final draft.

5. Let G be an arbitrary graph with at least 2 vertices. We construct a graph H by adding two vertices x and y to G, with every possible edge between vertices of G and x, y.

   For example (and this is **just** an example), if G is the graph below on the left (with 2 connected components), then H will be the graph below on the right:

   Prove that H will never have any cut vertices, no matter what graph G we start with.

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 5.