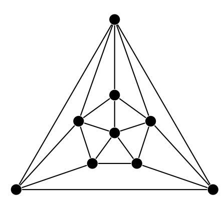
Graph Theory Homework 5

Mikhail Lavrov

due Friday, October 22, 2021

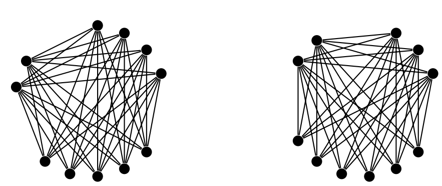
1 Short answer

1. Give an example of a graph G which is 3-regular and has:
   (a) $\kappa(G) = 1$.
   (b) $\kappa(G) = 2$.
   (c) $\kappa(G) = 3$.
2. Let J_{10} be the graph below. *(Irrelevant trivia: this is the skeleton graph of the* 10^{th} *Johnson solid, the gyroelongated square pyramid.)*

   Find a vertex v in J_{10} such that $J_{10} - v$ (the 8-vertex graph obtained by deleting v) is an Eulerian graph.
3. A complete tripartite graph is formed by taking three groups of vertices A, B, and C, then adding an edge between every pair of vertices in **different** groups. We write $K_{a,b,c}$ for the complete tripartite graph with $|A| = a$, $|B| = b$, and $|C| = c$.

   For example, below are diagrams of $K_{2,4,5}$ (left) and $K_{2,3,6}$ (right).

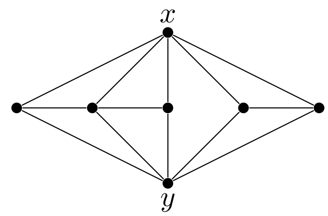
   

   (a) One of these graphs is Hamiltonian. Find a Hamiltonian cycle in that graph.
   (b) The other of these graphs is not Hamiltonian. Give a reason why it does not have a Hamiltonian cycle.

2 Proof

4. Let G be an arbitrary graph with at least 2 vertices. We construct a graph H by adding two vertices x and y to G, with every possible edge between vertices of G and x, y.

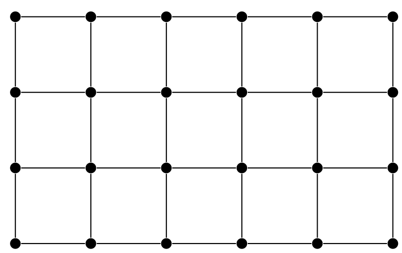
   For example (and this is **just** an example), if G is the graph below on the left (with 2 connected components), then H will be the graph below on the right:

   Prove that H will never have any cut vertices, no matter what graph G we start with.

   You have already written a rough draft of this solution; now, write a final draft.

5. Let G be the graph below: the "4×6 grid graph". An Eulerian tour in G would be a closed walk that uses every edge *exactly* once, but G doesn't have one of those.

   Find a closed walk in G that uses every edge *at most* once, and is as long as possible (it uses as many edges as possible). Prove that your solution is optimal.

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 5.