Probability Theory Homework 4

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due Friday, October 8, 2021

1. Three people named A, B, and C compete in a race. First place wins \$100, second and third place each win \$50.

   Let $\mathbf{W}$ be the dollar amount won by person A.

   (a) If the sample space is all 6 permutations of A, B, and C (in order that they finish the race), write down $\mathbf{W}$'s technical definition, as a function from the sample space to the real numbers. (That is, give $\mathbf{W}$'s value on each outcome.)

   (b) Write down the probability mass function $P_{\mathbf{W}}$, by giving its values (probabilities between 0 and 1) on each element of the range of $\mathbf{W}$.

2. A random variable $\mathbf{X}$ has probability mass function (PMF)

$$P_{\mathbf{X}}(k) = \begin{cases} C & k = 1 \\ 2C & k = 2 \\ C & k = 3 \\ 2C & k = 5 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of C for which this is a valid PMF.

   (b) Find $\Pr[\mathbf{X} \le 3]$.

   (c) Find $\mathbb{E}[\mathbf{X}]$.

3. A game show has two rounds. In the first round, you have 10 questions, and you have a $\frac{1}{2}$ chance of getting each one right.

   In the second round, you are given questions one at a time. You have a $\frac{2}{3}$ chance of getting each one right, and you will be asked questions until you get one wrong.

   (a) Let $\mathbf{R}_1$ be the number of questions you answer correctly in the first round.

   This random variable follows a named distribution we've studied in class. Give that distribution and its parameters.

   (b) Let $\mathbf{R}_2$ be the total number of questions you *see* in the second round.

This random variable follows a named distribution we've studied in class. Give that distribution and its parameters.

(c) If you win \$1000 for each correct answer, then your total winnings are given by $1000R_1 + 1000(R_2 - 1)$. (Ask me why if you're not sure.)

Find the expected amount you win: $\mathbb{E}[1000R_1 + 1000(R_2 - 1)]$.

4. Suppose that there are 10 possible topics on an exam. You did not have a long time to study, so you are only prepared for 5 topics.

The exam has 6 questions on it, which are selected to test 6 *different* topics (but are equally likely to be about any set of 6 topics.)

(a) Let $\mathbf{P}$ be the number of questions on the exam that you are prepared for.

This random variable follows a named distribution we've studied in class. Give that distribution and its parameters.

(b) Suppose that you get 15 points for each question you are prepared for, and 5 points for each question you are not prepared for. Let $\mathbf{S}$ be your score on the exam. Write $\mathbf{S}$ as a function of $\mathbf{P}$.

5. Let $\mathbf{N} \sim \text{Poisson}(\frac{1}{2})$.

(a) Use LOTUS to write down an infinite sum for $\mathbb{E}[\mathbf{N}!]$.

(b) Evaluate this infinite sum. (If (a) is correct, this should be a familiar summation.)