Probability Theory Homework 8

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due Monday, December 6, 2021

1. A mixed random variable $\mathbf{M}$ is generated by the following rule: $\mathbf{M}_1$ is chosen from the Uniform$(-1, 1)$ distribution, $\mathbf{M}_2$ is chosen uniformly from the set $\{-1, 0, 1\}$, and finally $\mathbf{M}$ is chosen to be $\mathbf{M}_1$ with probability $\frac{1}{4}$ and $\mathbf{M}_2$ otherwise.

   Find the CDF of $\mathbf{M}$.

   (If you read textbook section 4.3.2 on your own, you can instead give the PDF of $\mathbf{M}$ in terms of the Dirac delta function.)

2. On lucky days, the time you have to wait for a bus is Exponential$(\frac{1}{5})$ minutes. On unlucky days, the time is instead Gamma$(2, \frac{1}{5})$. In general, half of all days are lucky.

   (a) What is the expected time until the bus arrives?

   (b) If you have been waiting for the bus for 5 minutes, and it's still not here yet, what is the probability that it's an unlucky day?

3. Let $\mathbf{U}$ and $\mathbf{V}$ be independent random variables, with the PDFs

$$f_{\mathbf{U}}(s) = \begin{cases} \frac{1}{s^2} & s \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_{\mathbf{V}}(t) = \begin{cases} \frac{3}{8}t^2 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   Find $\Pr[\mathbf{U} \leq \mathbf{V}]$.

4. Let $\mathbf{Z} \sim \text{Normal}(0, 1)$, and let $Q(1) = \Pr[\mathbf{Z} > 1]$. *(There is no closed form for $Q(1)$, but it is approximately 0.159.)*

   Find the conditional expectation $\mathbb{E}[\mathbf{Z} \mid \mathbf{Z} > 1]$ in terms of $Q(1)$.

5. *(Not for credit, and don't write this up, but this is the homework problem I'd assign for the material from the lecture on Monday, December 6.)*

   A broken gumball vending machine spontaneously spits out gumballs of different colors. Over a short time scale, we can model this as a Poisson process which independently produces 0.1 red gumballs per second, 0.2 blue gumballs per second, and 0.3 yellow gumballs per second.

   Identify the distribution of the following quantities:

   (a) The number of blue gumballs produced in 1 minute.

   (b) The time until 5 yellow gumballs are produced.

   (c) The time interval between two consecutive gumballs produced (of any color).

   (d) The number of red gumballs produced after 10 total gumballs of all colors have been produced.