Linear Programming Homework #5

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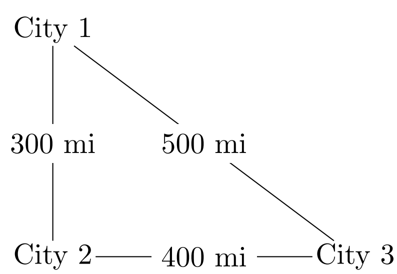
1. A roadside assistance company has an office in every city. Each office has an *assistance radius* within which they provide aid to cars that are in trouble on a highway.

   Some pairs of cities are connected by highways (each highway has a known length). Whenever a highway joins one city to another, the assistance radiuses (radii?) of the two cities must be large enough to cover the entire length of the highway.

   As long as this condition is met for every highway, the company wants the sum of all the assistance radii (radiuses?) to be as small as possible.

   (a) Write (but do not solve) a linear program to choose the assistance radius for every city, if there are three cities and three highways between them with the following lengths:

   (b) Suppose that we are given the following information: n, the total number of cities; H, a set of pairs (i, j) such that there is a highway between city i and city j; and a length ℓ_{ij} for every pair (i, j) in H telling us how long the highway is.

   Write down a general linear program by filling in the template below:

$$\begin{array}{lrcll} \text{minimize} & \sum_{i=1}^{n} \underline{\hspace{2cm}} & & & \\ \text{subject to} & \underline{\hspace{2cm}} & \geq & \underline{\hspace{2cm}} & \text{for all } (i,j) \in H \\ & \underline{\hspace{2cm}} & \geq & 0 & \end{array}$$

2. Consider the linear program

$$\begin{array}{ll} \underset{x_1,\dots,x_6 \in \mathbb{R}}{\text{maximize}} & x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \\ \text{subject to} & x_1 + 3x_2 + 2x_3 + 3x_4 + 3x_5 + 4x_6 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

(a) Write down the dual program. (Since the primal program has 6 variables and 1 constraint, the dual should have 1 variable and 6 constraints.)

(b) Solve the dual program, finding the optimal value of its only variable. (You will not have to use the simplex method here; you should be able to do this by looking, maybe after simplifying the constraints a little.)

(c) Use complementary slackness to find an optimal solution to the primal program.

3. Solve the following linear program using the dual two-phase simplex method:

$$\begin{array}{rrcl} \underset{x,y\in\mathbb{R}}{\text{maximize}} & 3x - y & & \\ \text{subject to} & 2x - y & \leq & -1 \\ & -x + y & \leq & 2 \\ & x, y & \geq & 0 \end{array}$$

4. Here is a linear program and an optimal dictionary:

$$\begin{array}{rrcl} \underset{x,y\in\mathbb{R}}{\text{maximize}} & 5x + 4y & & \\ \text{subject to} & x & \leq & 10 \\ & y & \leq & 10 \\ & x + 2y & \leq & 26 \\ & x, y & \geq & 0 \end{array} \qquad \begin{array}{rcl} \zeta & = & 82 - 3w_1 - 2w_3 \\ \hline x & = & 10 - w_1 \\ w_2 & = & 2 - \frac{1}{2}w_1 + \frac{1}{2}w_3 \\ y & = & 8 + \frac{1}{2}w_1 - \frac{1}{2}w_3 \end{array}$$

For small values of δ, describe the effect of the following changes on the objective function and on the optimal solution (x, y):

(a) The objective function changes from $5x + 4y$ to $(5 + \delta)x + 4y$.

(b) The upper bound on x changes from $x \leq 10$ to $x \leq 10 + \delta$.

(c) The upper bound on y changes from $y \leq 10$ to $y \leq 10 + \delta$.

5. Consider the matrix game with payoffs given by the table below:

	Bob: 1	Bob: 2	Bob: 3
Alice: 1	$(-1, 1)$	$(0, 0)$	$(2, -2)$
Alice: 2	$(1, -1)$	$(t, -t)$	$(3, -3)$
Alice: 3	$(0, 0)$	$(-1, 1)$	$(-2, 2)$

(a) Is this game a zero-sum game?

(b) For which values of t will "Alice picks 2, Bob picks 2" be a saddle point for this game?

(c) There is another potential saddle point for this game. What is it, and for what values of t is it a saddle point?