Linear Programming Homework #7

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due Friday, November 18, 2022

1. Suppose that you've solved the linear program given below on the left, and gotten the optimal dictionary on the right, with optimal solution $(x, y) = (\frac{4}{3}, \frac{13}{3})$:

$$\begin{array}{ll} \underset{x,y\in\mathbb{R}}{\text{maximize}} & 5x + 3y \\ \text{subject to} & -x + y \leq 3 \\ & 2x + y \leq 7 \\ & x - 2y \leq 0 \\ & x, y \geq 0 \end{array} \qquad \begin{array}{r} \zeta = \frac{59}{3} - \frac{1}{3}w_1 - \frac{8}{3}w_2 \\ \hline x = \frac{4}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 \\ y = \frac{13}{3} - \frac{2}{3}w_1 - \frac{1}{3}w_2 \\ w_3 = \frac{22}{3} - \frac{5}{3}w_1 - \frac{1}{3}w_2 \end{array}$$

   But then you're told that the problem was missing a constraint: the variable x must be an integer!

   (a) Adding the constraint $x \leq 1$ to your linear program, use the dual simplex method to continue from the dictionary given above and reach a new optimal solution.

   (b) Adding the constraint $x \geq 2$ to your linear program *instead*, use the dual simplex method to continue from the dictionary given above and reach a new optimal solution.

   (c) Given your answers to parts (a) and (b), what can you conclude about the optimal solution to the original problem, given that x must be an integer? Briefly explain why your conclusion makes sense.

2. There are sixteen composite numbers between 30 and 50:

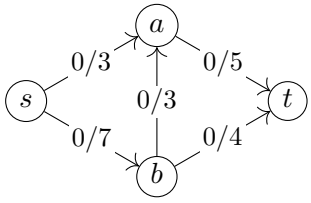
$$30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 49, 50.$$

   For each one of them, we want to pick a prime factor: for example, for 30, we could pick 2, 3, or 5.

   Prove that this cannot be done without picking the same prime factor for two of the numbers.

   (Hint: one possible solution goes back to our discussion of Hall's theorem.)

3. Consider the network given below:



(a) Write down the linear program for the max-flow problem in this network.

(b) Suppose that there is a fixed cost to using arc (b, a): if you send any positive amount of flow along this arc, you have to make a payment that's equivalent to reducing the value of the flow by 1. You do not have to make a payment if you do not use the arc, and the payment does not scale with how much flow you send along this arc.

Use a **integer variable** to modify your linear program for part (a) to take this fixed cost into account.

4. You have a collection of textbooks on linear programming that you would like to pack up in boxes. There are 10 introductory books that are $1.5''$ thick, and 10 advanced books that are $2.5''$ thick. Each box can hold a single stack of books that is at most $6.5''$ total.

Set up an integer linear program for packing your books in as few boxes as possible, using the configuration-LP technique from class. Do not solve.