Math 2390 Homework 5

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1 Short answer

1. Suppose we want to prove: "For all natural numbers n, if n is even, then $2^n - 1$ is divisible by 3." Classify each of the following initial assumptions as either a mistake, or the beginning of a direct proof, proof by contrapositive, or proof by contradiction.
   (a) Suppose that n is even.
   (b) Suppose that $2^n - 1$ is divisible by 3.
   (c) Suppose that n is odd, and that $2^n - 1$ is divisible by 3.
   (d) Suppose that n is even, and that $2^n - 1$ is not divisible by 3.
2. Disprove the claim: "For all natural numbers n, the number $n^2 - 3n + 13$ is prime."
3. Describe what a counterexample to each of these claims would (hypothetically) be.
   (a) For all real numbers α, there is a natural number n such that αn is within 0.0001 of an integer.
   (b) You can get from any Wikipedia article to any other by going through at most 6 links in the text of the article.
   (c) Someone in the class will get all the problems on this homework assignment right.
   (d) For all $\alpha \in \mathbb{R}$, if the sum $\sum_{n=1}^{\infty} n^{-\alpha}$ converges, then so does the integral $\int_1^{\infty} x^{-\alpha}\,dx$.

2 Proof

4. *For this problem, revise the rough draft you wrote for the previous assignment, based on my feedback. The result will be graded on correctness and clarity.*

   Let A and B be arbitrary sets. Prove that if $A \times A \subseteq B \times B$, then $A \subseteq B$.

5. *For this problem, write a rough draft of a proof; any reasonable attempt will be given full credit. I will give you feedback, and you will write a final draft on the next homework assignment.*

   Some background: a **Steiner triple system of order** n is a set of 3-element subsets (triples) of $\{1,2,\ldots,n\}$ such that for any pair $i,j \in \{1,2,\ldots,n\}$ with $i \neq j$, there is exactly one triple containing both i and j. For example,

   $$S = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\}$$

   is a Steiner triple system of order 7.

   These do not exist for all n. In particular, we can prove the following condition: for a Steiner triple of order n to exist, $\binom{n}{2} = \frac{n(n-1)}{2}$ must be divisible by 3. That's because there are $\binom{n}{2}$ pairs i,j, with each pair contained in exactly one triple—and each triple contains 3 pairs.

   In this problem, you'll look at what this does (and does not) tell us about n. Prove one of the following two statements, and disprove the other:

   - For all integers $n > 3$, if $\binom{n}{2}$ is divisible by 3, then $n \equiv 1 \pmod 6$.
   - For all integers $n > 3$, if $n \equiv 1 \pmod 6$, then $\binom{n}{2}$ is divisible by 3.