Math 2390 Homework 6

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1 Short answer

1. Suppose we want to argue, by using the definitions and axioms from our lecture on Peano arithmetic, that if $n+1=2$ for some $n \in \mathbb{N}_0$, then $n=1$. Give a justification for each of steps in the proof: each justification is either an axiom or a definition.

   We start by assuming that $n+1=2$, where $n \in \mathbb{N}_0$.

   (a) Therefore $n+s(0)=s(s(0))$, because ...

   (b) Therefore $s(n+0)=s(s(0))$, because ...

   (c) Therefore $s(n)=s(s(0))$, because ...

   (d) Therefore $n=s(0)$, because ...

   (e) Therefore $n=1$, because ...

2. A sequence q_n is defined by $q_1=1$ and $q_{n+1}=2q_n+2^n$ for all $n \ge 1$.

   Suppose you want to prove the formula $q_n = n \cdot 2^{n-1}$ by induction on n. In your induction step, what algebraic identity will you have to prove?

   (You don't actually need to prove anything or check any identities to answer this problem; just give the algebraic identity that the proof woul need.)

3. The Fibonacci numbers, which we have discussed in class, are a sequence defined by the rules $F_1=1$, $F_2=1$, and $F_n=F_{n-1}+F_{n-2}$ for $n \ge 2$.

   This problem is about the quantity $G_n = F_{n+2}F_{n+1} - F_{n+3}F_n$.

   (a) Compute G_n for a few values of n and make a guess about what it is in general.

   (b) Write down the following two expressions:

   - G_{n+1}, but with F_{n+4} rewritten in terms of F_{n+3} and F_{n+2};
   - G_n, but with F_n rewritten in terms of F_{n+1} and F_{n+2}.

Now both expressions are written entirely in terms of F_{n+1}, F_{n+2}, and F_{n+3}, which helps see the connection between them.

(c) What is the connection between the results you get, and how could you use it to prove the guess you made in part (a)?

2 Proof

4. *For this problem, revise the rough draft you wrote for the previous assignment, based on my feedback. The result will be graded on correctness and clarity.*

   Prove one of the following two statements, and disprove the other:

   - For all integers $n > 3$, if $\binom{n}{2}$ is divisible by 3, then $n \equiv 1 \pmod 6$.
   - For all integers $n > 3$, if $n \equiv 1 \pmod 6$, then $\binom{n}{2}$ is divisible by 3.

5. *For this problem, write a rough draft of a proof; any reasonable attempt will be given full credit. I will give you feedback, and you will write a final draft on the next homework assignment.*

   Prove by induction on n that for all $n \in \mathbb{N}$,

   $$1^4 + 2^4 + \cdots + (n-1)^4 < \frac{n^5}{5}.$$

   (When $n = 1$, the sum on the left-hand side is equal to 0, since there are no terms to sum.)