Math 2390 Homework 7

Mikhail Lavrov

due Friday, November 18, 2022

1 Short answer

1. Here is an *incorrect* proof of a *false* claim.

   False claim. For all $n \in \mathbb{N}$, $1 + 2 + 4 + \cdots + 2^n = 2^{n+1}$.

   "Proof". We induct on n. Suppose that the statement holds when $n = k$: $1 + 2 + 4 + \cdots + 2^k = 2^{k+1}$. If we add 2^{k+1} to both sides, we get

   $$1 + 2 + 4 + \cdots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} = 2^{k+2}$$

   so the statement continues to hold when $n = k + 1$. By induction, it holds for all n.

   Answer the following questions briefly (a sentence should be enough for each).

   (a) Why is the proof wrong?

   (b) Why is the claim false?

   (c) What is the difference between the questions in (a) and (b)?

2. For $a, b \in \mathbb{N}$, let $a \sim b$ if a and b have a nontrivial common divisor: in other words, if there is some integer $d > 1$ such that both a and b are divisible by d.

   For example, $15 \sim 25$ (since $5 \mid 15$ and $5 \mid 25$) but $15 \not\sim 16$ (since the only common divisor of 15 and 16 is 1).

   (a) Is $\sim$ reflexive? If not, find a counterexample.

   (b) Is $\sim$ symmetric? If not, find a counterexample.

   (c) Is $\sim$ transitive? If not, find a counterexample.

2 Proof

3. *This is the first time you're seeing this problem, but I'm not asking you to create your own proof from scratch, just to rewrite this one.*

   What follows is a proof of one concrete case of the division algorithm, by strong induction. Rewrite it as a proof by smallest counterexample.

   Claim. Every $n \in \mathbb{N}$ can be written as $n = 5q + r$, where $q, r \in \mathbb{Z}$ and $0 \le r < 5$.

   Proof. We induct on n. We begin by proving the claim for $n = 1, 2, 3, 4, 5$. These can be written as $5 \cdot 0 + 1$, $5 \cdot 0 + 2$, $5 \cdot 0 + 3$, $5 \cdot 0 + 4$, and $5 \cdot 1 + 0$, respectively.

   Now take an arbitrary $n \in \mathbb{N}$ such that $n > 5$, and assume that the claim is true for all natural numbers less than n.

   In particular, since $n > 5$, $n - 5 \in \mathbb{N}$, and the claim is true for $n - 5$. Therefore there are some $q, r \in \mathbb{Z}$ with $0 \le r < 5$ such that $n - 5 = 5q + r$. Then $n = 5(q+1) + r$, so n can also be written in the required form.

   By induction, the claim holds for all natural numbers. $\square$

4. *For this problem, revise the rough draft you wrote for the previous assignment, based on my feedback. The result will be graded on correctness and clarity.*

   Prove by induction on n that for all $n \in \mathbb{N}$,

   $$1^4 + 2^4 + \cdots + (n-1)^4 < \frac{n^5}{5}.$$

   (When $n = 1$, the sum on the left-hand side is equal to 0, since there are no terms to sum.)

5. *For this problem, write a rough draft of a proof; any reasonable attempt will be given full credit. I will give you feedback, and you will write a final draft on the next homework assignment.*

   Define a sequence (x_n) by taking $x_1 = \frac{1}{4}$ and $x_{n+1} = 2x_n(1 - x_n)$. This sequence will get approach $\frac{1}{2}$ very quickly; you can convince yourself of this by trying it out with a calculator.

   Prove that for all $n \in \mathbb{N}$,

   $$\frac{1}{2} - \frac{1}{2^{n+1}} \le x_n < \frac{1}{2}.$$

   You can do this however you like, but two reasonable approaches are (1) try to prove this inequality by induction, or (2) prove a formula for x_n, and verify that it satisfies the inequality.

   Note that this inequality considerably underestimates how close x_n will get to $\frac{1}{2}$.