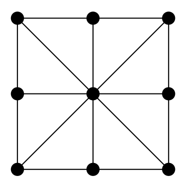
Math 2390 Homework 8

Mikhail Lavrov

due Friday, December 2, 2022

1 Short answer

1. In the diagram below, there are 9 points and 8 lines:



   (a) Let $\diamond$ be the relation between the *points* in the diagram, where $A \diamond B$ if one of the 8 lines passes through both A and B.

   Explain why $\diamond$ is not an equivalence relation.

   (b) Let $\circ$ be the relation between the *lines* in the diagram, where $\ell_1 \circ \ell_2$ if either $\ell_1 = \ell_2$ or else ℓ_1 and ℓ_2 have no points in common.

   This is an equivalence relation! Find the equivalence classes of $\circ$.

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by the following rule:

$$f(x) = \begin{cases} x+6 & x \equiv 0 \pmod 3 \\ x+5 & x \equiv 1 \pmod 3 \\ x+8 & x \equiv 2 \pmod 3 \end{cases}$$

For example, to find $f(100)$, we determine first that $100 = 3 \cdot 33 + 1$ and therefore $100 \equiv 1 \pmod 3$; therefore $f(100) = 105$.

   (a) Give an example to show that f is not injective.

   (b) Give an example to show that f is not surjective.

   (c) Change *one* of the numbers 6, 5, and 8 to turn f into a bijection.

3. Let $g\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ be given by the formula

$$g(m,n) = (m+n, m+2n)$$

For example, $g(1,2) = (1+2, 1+2\cdot 2) = (3,5)$.

Find a formula for the inverse of g.

2 Proof

4. *For this problem, revise the rough draft you wrote for the previous assignment, based on my feedback. The result will be graded on correctness and clarity.*

   Define a sequence (x_n) by taking $x_1 = \frac{1}{4}$ and $x_{n+1} = 2x_n(1-x_n)$. This sequence will get approach $\frac{1}{2}$ very quickly; you can convince yourself of this by trying it out with a calculator.

   Prove that for all $n \in \mathbb{N}$,
   $$\frac{1}{2} - \frac{1}{2^{n+1}} \le x_n < \frac{1}{2}.$$
   You can do this however you like, but two reasonable approaches are (1) try to prove this inequality by induction, or (2) prove a formula for x_n, and verify that it satisfies the inequality.

   Note that this inequality considerably underestimates how close x_n will get to $\frac{1}{2}$.

5. *This problem is just for practice! I will not grade you on it, but if you submit it with your homework, I will give you feedback on your proof.*

   Let $\sim$ be the relation defined on $\mathbb{R}$ by the condition that $x \sim y$ if $x^3 - y^3 = 13(x-y)$. For example, $1 \sim 3$, because $1^3 - 3^3 = 1 - 27 = -26$, which is 13 times $1-3$.

   Prove that $\sim$ is an equivalence relation.

3 One final note

If more than half of the class fills out student evaluations before the final, then I will wear a turkey hat for the duration of the final exam.