Calculus IV Homework 1

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due Friday, August 23, 2024

0. On Wednesday, August 21, I will have office hours 12pm–5pm, in case you have questions about the homework assignment (or anything else).

   Together with your answers to this homework assignment, let me know when you'd like me to have permanent office hours this semester—if you have a preference. Or, feel free to vote with your feet by showing up to office hours!

   (My theory is that I will have regular office hours on Wednesday, and possibly extra office hours on Monday the week of an exam.)

1. Set up each of the following integrals, using cylindrical coordinates. You don't have to evaluate the integrals.

   (a) The integral for the volume of a solid sphere of radius 1 centered at the origin. (Yes, it would make more sense to use spherical coordinates. I'm being unreasonable.)

   (b) The integral of $x + y + z$ over the interior of a cone whose base is a disk of radius 1 in the xy-plane centered at the origin, and whose tip is the point $(0, 0, 2)$.

   (c) The integral of $x^2+y^2+z^2$ over the half-cylinder bounded by the inequalities $x^2+y^2 \leq 1$, $0 \leq z \leq 3$, and $y \geq 0$.

2. Set up each of the following integrals, using spherical coordinates. You don't have to evaluate the integrals.

   (a) The integral for the volume of the portion of the solid sphere of radius 2 centered at the origin with $x \geq 0$, $y \geq 0$, and $z \geq 0$.

   (b) The integral of $x^2 + y^2$ over a hollow spherical shell of thickness 1 and inner radius 3 centered at the origin.

   (c) The integral for the volume of a sphere of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$. (Yes, it would make more sense to center the sphere at the origin. I'm being unreasonable again.)

3. Perform each of the substitutions given below. You don't have to evaluate the resulting integrals. ~~For the sake of variety, use the Jacobian at least once, and use the $\mathrm{d}x \wedge \mathrm{d}y$ method at least once.~~

   (a) Rewrite the integral

$$\int_{x=2}^{5} \int_{y=-x}^{2-x} \sqrt{x^2 + 1}\, \mathrm{d}y\, \mathrm{d}x$$

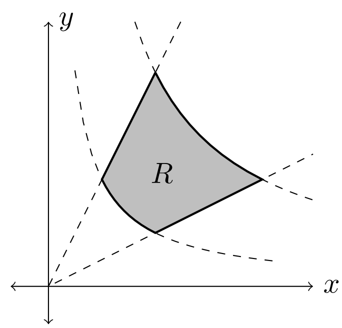
using the substitution $u = x^2 + 1$, $v = x + y$.

(b) Rewrite the sum of two integrals

$$\int_{x=-1}^{0} \int_{y=-x-1}^{x+1} (x+y)\,\mathrm{d}y\,\mathrm{d}x + \int_{x=0}^{1} \int_{y=x-1}^{1-x} (x+y)\,\mathrm{d}y\,\mathrm{d}x$$

as a single integral using the substitution $u = x + y$, $v = x - y$. (To see what's going on, it may help to draw a diagram.)

(c) Let R be the region in the first quadrant of the xy-plan consisting of those points where $1 \le xy \le 4$ and $\frac{1}{2} \le \frac{x}{y} \le 2$, shown below:



Rewrite the integral $\iint_R \mathrm{d}y\,\mathrm{d}x$ using the substitution $u = xy$, $v = \frac{x}{y}$.

4. Let S be the solid hemisphere which is the portion of the sphere of radius 1 centered at $(0,0,0)$ that lies above the plane $z = 0$. Find the average value of $f(x,y,z) = z^2$ over S.

5. Compute the integral

$$\int_{x=1}^{7} \int_{y=x-1}^{x+1} \int_{z=x}^{3x} \frac{y-1}{2x}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

by first performing the substitution $u = \frac{x-1}{2}$, $v = x - y$, $w = \frac{z}{x}$.