Calculus IV Homework 7

Mikhail Lavrov

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1. Find the curl of the following vector fields:
   (a) $\mathbf{F} = \cos(e^x)\,\mathbf{i} + \sqrt{y^3 + y^6 + y^7}\,\mathbf{j} + z^{1+\tan^2 z}\,\mathbf{k}$.
   (b) $\mathbf{F} = yz\,\mathbf{i} - 2xz\,\mathbf{j} + xy\,\mathbf{k}$.
   (c) $\mathbf{F} = xy\cos z\,\mathbf{i} + xy\cos z\,\mathbf{j} + xy\cos z\,\mathbf{k}$.

2. Each part of this problem gives you a surface S with a parameterization. Use that surface parameterization to find four parameterizations $\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t), \mathbf{r}_4(t)$ for the boundary of S. Then, list which (if any) of these boundaries cancel with each other, and which (if any) are trivial; briefly describe the remaining boundaries geometrically.

(example) The cylinder parameterized by $\mathbf{r}(u,v) = (\cos u, \sin u, v)$, where $(u,v) \in [0,2\pi] \times [0,3]$.

Solution: the four parameterizations we get for the boundary are are

$$\begin{aligned}
\mathbf{r}_1(t) &= \mathbf{r}(t,0) = (\cos t, \sin t, 0) && t \in [0,2\pi] \\
\mathbf{r}_2(t) &= \mathbf{r}(2\pi,t) = (1,0,t) && t \in [0,3] \\
\mathbf{r}_3(t) &= \mathbf{r}(-t,3) = (\cos t, -\sin t, 3) && t \in [-2\pi,0] \\
\mathbf{r}_4(t) &= \mathbf{r}(0,-t) = (1,0,-t) && t \in [-3,0]
\end{aligned}$$

Here, $\mathbf{r}_1(t)$ and $\mathbf{r}_3(t)$ are circles around the top and bottom edges of the cylinder, and $\mathbf{r}_2(t)$ and $\mathbf{r}_4(t)$ cancel with each other.

   (a) The portion of the unit sphere with $x \ge 0$, $y \ge 0$, and $z \ge 0$, which is parameterized by $\mathbf{r}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$, where $(u,v) \in [0,\pi/2] \times [0,\pi/2]$.
   (b) The portion of the paraboloid $x = y^2 + z^2$ with $0 \le x \le 1$, which is parameterized by $\mathbf{r}(u,v) = (u^2, u\cos v, u\sin v)$, where $(u,v) \in [0,1] \times [0,2\pi]$.

3. Let S be the portion of the cone with equation $z^2 = x^2 + y^2$ which lies between the planes $z = 1$ and $z = 2$, oriented so that the normal vectors point inward and upward.
   (a) Give S a parameterization of the form $\mathbf{r}(u,v)$, where $(u,v) \in [a,b] \times [c,d]$, making sure to be consistent with the orientation of S we want.
   (b) As you did in problem 2, use this parameterization to find the boundaries of S.
   (c) Let $\mathbf{F}$ be the vector field $xy\,\mathbf{i} + z^2\,\mathbf{j} - (x - z)\,\mathbf{k}$. Stokes' theorem relates the curl integral of $\mathbf{F}$ over S to the circulation integral of $\mathbf{F}$ along the boundary of S. The surface in this

problem has two boundaries which we can call C and C'. This means that in this case, Stokes' theorem tells us that

$$\iint_S \_\_\_\_\, \mathrm{d}y \wedge \mathrm{d}z + \_\_\_\_\, \mathrm{d}z \wedge \mathrm{d}x + \_\_\_\_\, \mathrm{d}x \wedge \mathrm{d}y = \int_C \_\_\_\_\, \mathrm{d}x + \_\_\_\_\, \mathrm{d}y + \_\_\_\_\, \mathrm{d}z$$
$$+ \int_{C'} \_\_\_\_\, \mathrm{d}x + \_\_\_\_\, \mathrm{d}y + \_\_\_\_\, \mathrm{d}z.$$

Fill in the blanks, and describe C and C' geometrically. You do not have to evaluate the integrals.

4. Let S be the rectangle in the plane $y + z = 1$ which has corners at $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, and $(0, 1, 0)$, oriented so that the normal vectors point away from the origin.

   Find the curl integral of $\mathbf{F} = xyz\,\mathbf{i} - xz\,\mathbf{j} + y^2\,\mathbf{k}$ over S.

5. The cone $z^2 = x^2 + y^2$ and the plane $x = 2z - 3$ intersect in an ellipse C which has parameterization $\mathbf{r}(t) = (1 + 2\cos t, \sqrt{3}\sin t, 2 + \cos t)$, $t \in [0, 2\pi]$.

   (a) Find the circulation of $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$ around C (using the parameterization above). We covered circulation integrals much earlier this semester, but you should still remember how to do them!

   (b) The portion of the cone above the xy-plane but below the plane $x = 2z - 3$ is a surface with boundary C. One way to take a flux integral across this surface is to treat it as a surface defined by the implicit equation $x^2 + y^2 - z^2 = 0$ above a region R in the xy-plane.[1]

   Write down a curl integral over S which is equal to the circulation of $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$ around C, and transform it into a double integral over R with respect to x and y.

   (c) Use parts (a) and (b) to determine the area of R.

[1] More precisely, R happens to be the interior of another ellipse: it is the region defined by $3(x - 1)^2 + 4y^2 \le 12$.