Calculus IV Homework 8

Mikhail Lavrov

due Friday, November 22, 2024

1 Course evaluations

Course evaluations for this class (and other classes) should now be accessible on D2L. The deadline for filling out course evaluations is December 2nd: after classes are over, but before our final exam on December 5th.

These are very useful to me in deciding how to teach this class in the future. (Comments are especially helpful, but even if you don't leave those, numerical ratings are helpful, and they're more reliable if more of you give them.) The evaluations are anonymous, and are not released until after final grades are due, so there are no drawbacks to being honest.

As an incentive, if at least 50% of the class fills out a course evaluation, I will wear a funny hat for the final exam.

2 The divergence theorem

1. Find the outward flux of $\mathbf{F} = (x - 2y)\,\mathbf{i} - (3z + 4x)\,\mathbf{j} + (5y + 6z)\,\mathbf{k}$ across the surface of the cube of side length 2 in $\mathbb{R}^3$ given by the inequalities $|x| \le 1$, $|y| \le 1$, and $|z| \le 1$.

2. Let S_1 be the portion of the unit sphere with $z \ge 0$, oriented with normal vector pointing up and out of the sphere.

   Let S_2 be the portion of the unit sphere with $z \le 0$, oriented with normal vector pointing up and into the sphere.

   Finally, let $\mathbf{F} = 2y\,\mathbf{i} + 2z\,\mathbf{j} + 2x\,\mathbf{k}$.

   (a) Find a vector field $\mathbf{G}$ such that the curl $\nabla \times \mathbf{G}$ is equal to $\mathbf{F}$. *(There are infinitely many possibilities, but I've set the problem up so that some are particularly easy to find.)*

   (b) Use Stokes' theorem to explain why the flux integral of $\mathbf{F}$ across S_1 is equal to the flux integral of $\mathbf{F}$ across S_2.

   (c) Use the divergence theorem to explain why the flux integral of $\mathbf{F}$ across S_1 is equal to the flux integral of $\mathbf{F}$ across S_2.

3. Let D be the cone of height 1 above the unit circle in the xy-plane: the cone described by

$$\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1 \text{ and } 0 \leq z \leq 1-\sqrt{x^2+y^2}.\}$$

Let $\mathbf{F} = (x+y)^2\,\mathbf{i} + z^2\,\mathbf{k}$.

(a) The cone D has two boundaries: a flat disk in the xy-plane, and a lateral boundary that lies on the graph of $z = 1-\sqrt{x^2+y^2}$.

Explain why the (outward) flux integral of $\mathbf{F}$ across the boundary in the xy-plane is 0.

(b) Use the divergence theorem to find the outward flux integral of $\mathbf{F}$ across the lateral boundary of the cone.

3 Review

4. Parameterize the following objects:

(a) The parallelogram with corners at $(0,0,0)$, $(0,1,1)$, $(2,2,1)$, and $(2,1,0)$.

(Find a paramerization with rectangular domain $(u,v) \in [a,b] \times [c,d]$.)

(b) A "washer" shape in the xy-plane bounded by the inequalities $1 \leq x^2+y^2 \leq 4$.

(Find a paramerization with rectangular domain $(u,v) \in [a,b] \times [c,d]$.)

(c) The curve where the cylinder $x^2+y^2=1$ intersects the plane $x+y+z=1$.

(This is a curve, so find a parameterization with one variable $t \in [a,b]$.)

5. (a) Describe the shape of the curve C parameterized by $\mathbf{r}(t) = (t\cos t, t\sin t, t)$ where $t \in [0, 10\pi]$.

(b) Find the gradient ∇f of the scalar function $f(x,y,z) = xe^{x+y+z}$.

(c) Find the value of the vector line integral of ∇f, the gradient field from part (b), along the curve C from part (a).