Graph Theory Homework 1

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due Friday, August 23, 2024

Office hours

0. On Wednesday, August 21, I will have office hours 12pm–5pm, in case you have questions about the homework assignment (or anything else).

   Together with your answers to this homework assignment, let me know when you'd like me to have permanent office hours this semester—if you have a preference. Or, feel free to vote with your feet by showing up to office hours!

   (My theory is that I will have regular office hours on Wednesday, and possibly extra office hours on Monday the week of an exam.)

1 Short answer

In this section, all I expect you to do is find the answer, possibly accompanied by a diagram or a one-sentence explanation if I ask for it.

1. Imagine a puzzle in which four coins are lined up in a row: two pennies and two dimes. In one step, you can swap two adjacent coins.

   (a) Let G_1 be the graph whose vertices are possible states of this puzzle, with an edge between states that are one step apart. *(Do not include "steps" that don't do anything because they swap two identical coins.)*

   Draw a diagram of G_1.

   (b) How many vertices and how many edges does G_1 have?

   (c) Find a cycle in G_1.

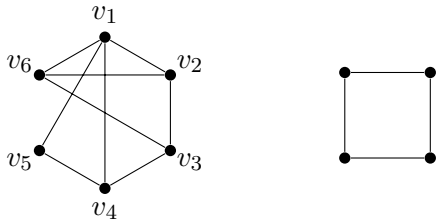
2. Let G_2 be the graph whose vertices are the numbers $1, 2, \ldots, 10$, with an edge between a and b if $|a - b| \geq 6$. (For example, vertex 3 is adjacent to vertex 10, because $|3 - 10| = 7$, but not to vertex 6, because $|3 - 6| = 3$.)

   (a) Draw a diagram of G_2.

   (b) What are the connected components of G_2?

   (c) There are two vertices x and y such that $G_2 - x$ and $G_2 - y$ have more connected components than G_2 does. What are those vertices?

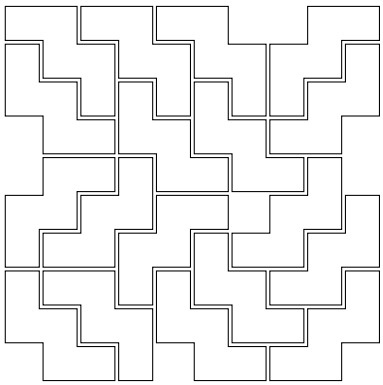
3. Let G_3 be the graph shown below on the left. Inside G_3, we want to find subgraphs that look like the cycle graph C_4 (shown on the right).



   (a) How many such subgraphs are there? List them all, by telling me which vertices and edges are part of each subgraph.

   (b) Which of the subgraphs you've found are induced?

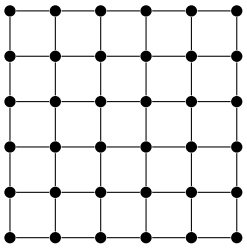
4. In class, we discussed the graph coloring problem, using a tiling of a 10×10 grid with zigzag tiles as an example. Here is a diagram of such a tiling. Find a way to color it using only 3 colors so that no two tiles that share a border have the same color.



2 Proof

In this section, you should write a proof. Write in complete sentences and justify your logic.

5. Let G be the 6×6 grid graph, shown below.



   Prove that the diameter of G is at most 10.

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 2.