Graph Theory Homework 2

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1 Short answer

1. Define Q_n, as in class, to be the graph with vertex set $\{0,1\}^n$ and an edge between any two vertices that differ in one position. Let G_n be the subgraph of Q_n induced by the vertices in which the total number of 1's is either one or two.

   (a) Draw a diagram of G_3 (it should have six vertices), and find its degree sequence (as a list of six numbers).

   (b) Find the degree sequence of G_4 (you do not have to draw a diagram).

   (c) How many edges will the graph G_n have, as a function of n?

2. Suppose that H is a bipartite graph. On one side of the bipartition, there are n vertices; their degrees are $1, 2, 3, \ldots, n$. On the other side of the bipartition, there are also n vertices; all of them have degree 4. What is n?

3. For each of the statements below, say whether it is true or false. If it is true, briefly explain why; if it is false, draw a counterexample.

   (a) If the minimum degree of a graph is odd, then the graph must have an even number of vertices.

   (b) If a graph is bipartite, then it must have an even number of vertices.

   (c) If a graph has minimum degree at least 1, then it must be connected.

   (d) If a graph is connected, it must have minimum degree at least 1.

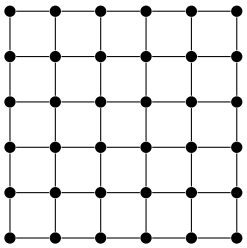
2 Proof

4. Prove the following by induction on n. For all $n \geq 5$, there exists a graph with n vertices and $2n - 4$ edges that has minimum degree 2 and maximum degree 4.

   (If you're having trouble, begin by omitting the "maximum degree 4" condition; then, see if you can find a way to modify your proof to include it.)

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 3.

5. Let G be the 6×6 grid graph, shown below.



Prove that the diameter of G is at most 10.

You have already written a rough draft of the solution; now, write a final draft.