Graph Theory Homework 5

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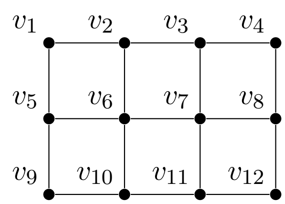
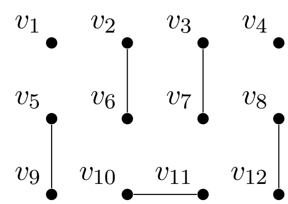
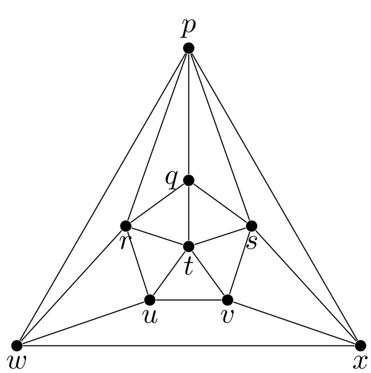
1 Short answer

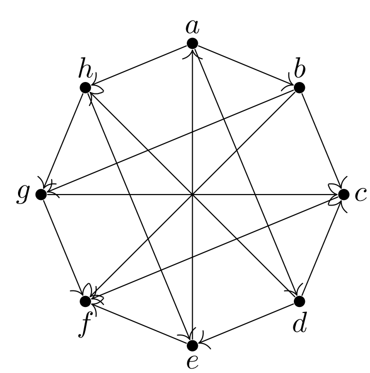
1. In the graph shown on the left below, find an augmenting path for the matching $M = \{v_2v_6, v_3v_7, v_5v_9, v_8v_{12}, v_{10}v_{11}\}$ (shown on the right). Then, augment M by that path to get a bigger matching.





2. Let J_{10} be the graph below on the left. *(Irrelevant trivia: this is the skeleton graph of the* 10^{th} *Johnson solid, the gyroelongated square pyramid.)* Find a vertex v in J_{10} such that deleting it produces an 8-vertex *Eulerian* graph.





3. In the directed graph shown above on the right, find an arc (u, v) such that if it is reversed (deleted and replaced by (v, u)), the result is acyclic. Give a topological ordering of the resulting directed graph after (u, v) is reversed.

   If the picture is unclear, the set of arcs is $\{(a, b), (a, d), (a, h), (b, c), (b, f), (b, g), (c, f), (d, c), (d, e), (d, h), (e, a), (e, f), (g, c), (g, f), (h, e), (h, g)\}$.

   (Hint: if you're not sure how to get started, try looking for cycles, or try finding the strongly connected components.)

2 Proof

4. In a directed graph, it is possible to start at a vertex, follow some of the edges, and then end up unable to return to where you started. In other words, it is possible that a directed graph contains an $x - y$ walk but not a $y - x$ walk for some vertices x and y.

   Let's call a vertex x **safe** if it's impossible to leave it and get lost: for every vertex y such that there is an $x - y$ walk, there is also a $y - x$ walk.

   Prove that every directed graph has at least one safe vertex.

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 5.

5. What is the largest possible size of a matching in a tree T with 60 vertices, 40 of which are leaves? You should prove both parts of the answer: if you say "the largest possible size is m", then you should give an example of a 60-vertex, 40-leaf tree with a matching of size m, *and* prove that there is no 60-vertex, 40-leaf tree with a matching of size $m + 1$ or more.

   You have already written a rough draft of the solution; now, write a final draft.