Probability Theory Homework 1

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due Friday, August 23, 2024

0. On Wednesday, August 21, I will have office hours 12pm–5pm, in case you have questions about the homework assignment (or anything else).

   Together with your answers to this homework assignment, let me know when you'd like me to have permanent office hours this semester—if you have a preference. Or, feel free to vote with your feet by showing up to office hours!

   (My theory is that I will have regular office hours on Wednesday, and possibly extra office hours on Monday the week of an exam.)

1. In a children's game, you take a turn by rolling a fair 6-sided die. If you roll a 1, then you get to roll a second time, and add the two rolls together. This determines how many spaces you get to move along a track.

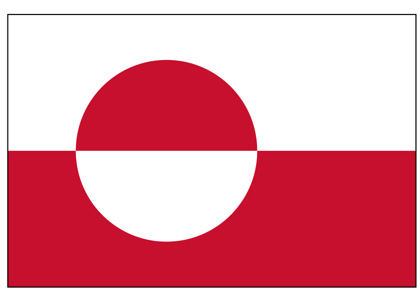
   (a) List all the outcomes in the sample space of the random experiment of rolling to see how many spaces you move.

   (b) Are we sampling uniformly from this sample space? Why or why not?

   (c) What is the probability that you move forward at least 5 spaces?

2. There are 25 prime numbers between 1 and 100. If you choose a random number from 1 to 100, what is the probability that it is either prime or even? (There is only one even prime number, and that is 2.)

3. The flag of Greenland is shown below:

   It is a 12×18 rectangle; if placed on a coordinate plane with bottom left corner at $(0, 0)$ and top right corner at $(18, 12)$, it is divided in half by a line at $y = 6$ and has a circle of radius 4 centered at $(7, 6)$.

The top half of the flag is white and the bottom half is red; within the circle, the two colors are swapped.

Suppose that a point on this flag is chosen uniformly at random.

(a) Find the probability that the chosen point is red.

(b) Find the probability that the chosen point is red **or** inside the circle. (As usual, "or" in mathematics includes the possibility that both things happen.)

4. You have an unfair coin which lands heads $\frac{2}{3}$ of the time, and tails $\frac{1}{3}$ of the time. You decide to flip the coin over and over until you get the same outcome two times in a row.

   (a) Describe the sample space of this experiment as a countably infinite set of outcomes, writing each outcome as a finite sequence like HTHTT. (Of course, you won't be able to write all infinitely many outcomes down, so just convey the idea of what they will be.)

   (b) Find the probability that you flip the coin exactly three times.

5. Suppose that you know that your friend was born sometime in the year 2003, but not which day. Given this state of ignorance, we can model your friend's birthday as being chosen uniformly at random from the 365 days of a non-leap year. (In reality, these are not entirely uniform, but they're pretty close, and we can ignore this effect.)

   (a) What is the probability that your friend was born in January?

   (b) Suppose you remember that your friend's birthday is on the 20^{th} (of some month). Conditioned on this fact, what is the probability that your friend was born in January?

   (c) Now suppose you remember that your friend's birthday is on the 30^{th} (of some month). Conditioned on this fact, what is the probability that your friend was born in January?