Probability Theory Homework 4

Mikhail Lavrov

due Friday, October 4, 2024

1. Each of the following random variables has a Binomial, Geometric, or Hypergeometric distribution. Identify the distribution, and give its parameters.

   (Note: we covered these briefly in class on 9/17 and will discuss them again. Section 3.1.5 of the textbook also has a summary of these and a couple others we'll include as time goes on. My notation for the Hypergeometric distribution differs slightly from the textbook's; I use n rather than k for the number of samples.)

   (a) You draw a hand of 5 cards from a standard 52-card deck. $\mathbf{A}$ is the number of aces you draw. (There are 4 aces in the deck.)

   (b) You are looking for a four-leaf clover in a clover field. The four-leaf mutation is quite rare, with an incidence rate of 0.1%. $\mathbf{C}$ is the number of clovers you will have to inspect before you find a four-leaf clover.

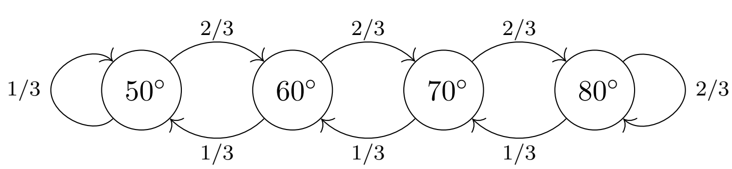
   (c) You receive many emails every day, but 90% of them are junk emails. (Let's assume that you don't have a spam filter to catch these junk emails.) Today, you receive 15 emails; $\mathbf{J}$ is the number of them that are junk.

2. A basketball player is practicing free throws, and has a $\frac{1}{2}$ chance of making the first shot. However, the basketball player is very unconfident, so every shot after that strongly depends on the result of the previous throw:

   - After a successful throw, the basketball player has a $\frac{1}{2}$ chance of making the next shot and a $\frac{1}{2}$ chance of missing.
   - After a miss, the basketball player has only a $\frac{1}{3}$ chance of making the next shot, and a $\frac{2}{3}$ chance of another miss.

   Give a table of the probability that the basketball player makes the k^{th} shot for $k = 1, 2, 3$.

3. The temperature (in Fahrenheit) in a certain region is always one of $\{50°, 60°, 70°, 80°\}$ and changes from day to day according to the following Markov chain:



(That is, every day, there is a $\frac{2}{3}$ chance that it gets warmer—except that when it's 80°, the temperature just stays at 80°. Every day, there is a $\frac{1}{3}$ chance that it gets colder—except that when it's 50°, the temperature just stays at 50°.)

(a) Solve for the limiting probabilities $\pi_{50}, \pi_{60}, \pi_{70}, \pi_{80}$ of having each temperature, in the long run.

(Hint: try solving for $\pi_{60}, \pi_{70}, \pi_{80}$ in terms of π_{50} first. Then determine what π_{50} must be.)

(b) Find the average temperature in the region.

4. A random variable $\mathbf{X}$ has range $R_{\mathbf{X}} = \{1, 2, 3, 4, 5, 6\}$ and probability mass function $P_{\mathbf{X}} \colon R_{\mathbf{X}} \to [0, 1]$ given by

$$P_{\mathbf{X}}(k) = \begin{cases} c & k \in \{1, 2\}, \\ 2c & k \in \{3, 4, 5, 6\}. \end{cases}$$

(Watch out for a common mistake: this piecewise definition says that $P_{\mathbf{X}}(1) = c$ and that $P_{\mathbf{X}}(2) = c$, not that $\Pr[\mathbf{X} \in \{1, 2\}] = c$.)

(a) Find the value of c for which this is a valid probability mass function.

(b) Find $\Pr[\mathbf{X} \geq 4]$.

(c) Find the expected value $\mathbb{E}[\mathbf{X}]$.