Probability Theory Homework 7

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1. We roll three fair dice; let $\mathbf{X}$ be the number of 1's rolled, and let $\mathbf{Y}$ be the number of 6's rolled. Find the covariance $\mathrm{Cov}[\mathbf{X}, \mathbf{Y}]$.

2. Consider the following experiment. We choose a uniformly random word from the sentence

   "The quick brown fox jumped over the lazy dog."

   Let $\mathbf{V}$ be the number of vowels and $\mathbf{C}$ be the number of consonants in the word chosen. (Here, we consider a, e, o, u, i, y to be vowels, and all other letters to be consonants.)

   (a) Give the joint probability mass function of $\mathbf{C}$ and $\mathbf{V}$, in a table.

   (b) Find the marginal distributions of $\mathbf{C}$ and $\mathbf{V}$.

   (c) Suppose that instead $\mathbf{C}$ and $\mathbf{V}$ were computed for a uniformly random word from the English dictionary. Do you expect the covariance $\mathrm{Cov}[\mathbf{C}, \mathbf{V}]$ to be positive or negative? Explain why.

3. Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables with the distributions $\mathbf{X} \sim \text{Geometric}(p = \frac{1}{3})$ and $\mathbf{Y} \sim \text{Geometric}(p = \frac{2}{3})$. Find $\Pr[\mathbf{X} = \mathbf{Y}]$.

4. For each of the random variables below, plot the CDF.

   (a) A random real number chosen uniformly from the interval $[-2, 2]$.

   (b) A discrete random variable equal to -1 with probability $\frac{2}{3}$ and to 1 with probability $\frac{1}{3}$.

   (c) A random real number chosen uniformly from the set $[-2, -1] \cup [1, 2]$: the union of two intervals with a gap between them.

5. A random variable $\mathbf{A}$ has the probability density function

$$f_{\mathbf{A}}(t) = \begin{cases} c \cdot (2 - t) & 0 \le t \le 2 \\ c \cdot (t - 3) & 3 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}$$

   for some constant c.

   (a) Find the value of c that makes $f_{\mathbf{A}}(t)$ a valid PDF.

   (b) Find the cumulative distribution function of $\mathbf{A}$.

   (c) Find $\Pr[1 \le \mathbf{A} \le 3]$.