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# Putnam practice #1: A variety of problems

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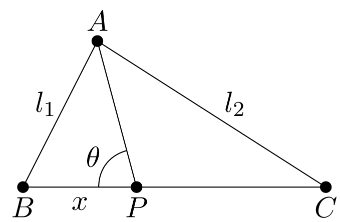
## 1 Problems

**UW Math Hour 2012.** Katniss is thinking of a positive integer less than 100: call it $x$. Peeta is allowed to pick any two positive integers $m$ and $n$, both less than 100, and Katniss will give him the greatest common divisor of $x + m$ and $n$. Peeta can do this up to seven times, after which he must name Katniss's number $x$, or he will die. Can Peeta ensure his survival?

**Mathcamp 2018.** Let $N$ be an integer whose decimal representation has the form $3 \ldots 36 \ldots 67$: some number of 3's, followed by some (possibly different) number of 6's, followed by a 7.

Prove that the digits of $N^2$, going in order from left to right, never decrease.

**VTRMC 2000/4.** In the diagram below, $l_1 = \overline{AB}$, $l_2 = \overline{AC}$, $x = \overline{BP}$, and $l = \overline{BC}$, where $\overline{XY}$ indicates the length of a segment $XY$. Prove that $l_2 - l_1 = \int_0^l \cos(\theta(x))\, \mathrm{d}x$.




**Putnam 2005/B1.** Find a nonzero polynomial $P(x, y)$, such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$.[1]

**Putnam 1996/A2.** Let $C_1$ and $C_2$ be circles whose centers are 10 units apart; $C_1$ has radius 1 and $C_2$ has radius 3. Find, with proof, the locus[2] of all points $M$ for which there exists points $X$ on $C_1$ and $Y$ on $C_2$ such that $M$ is the midpoint of the line segment $XY$.

**Putnam 2002/A4.** In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty $3 \times 3$ matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who wins and how?

**A. Shen, *Mathematical Induction.*** Suppose that you make a deal with the devil: whenever you want, you can trade a paper bill for any number of bills of lesser value. (For example, you can trade one \$10 bill for a million \$5 bills.) You keep all the cash you currently have, but you cannot earn more, other than by trading with the devil; in particular, you cannot receive change.

Each day, you need to spend at least \$1 to eat. Can you live forever?

[1] $\lfloor x \rfloor$ denotes the floor of $x$: the greatest integer less than or equal to $x$. For example, $\lfloor \pi \rfloor = 3$.
[2] "Locus" is a fancy geometry word for the set of all points satisfying some property.

## 2 Problem-solving tips

1. **Try small cases.** In many cases, you can solve the general problem by finding a pattern in a few small special cases. Even if that doesn't help, solving the problem in an easier case may give you ideas for how to tackle it in a harder case.
2. **Explore.** If a problem presents an unfamiliar setting, try playing around with it even if it doesn't feel like you're making immediate progress. Sometimes, becoming more comfortable with the details of the problem will let you spot a solution much more easily.
3. **Make connections.** Think about other similar problems you've solved in the past. Was there an idea that helped you there? Can you use the idea again to solve the problem you're facing now?
4. **Make conjectures.** Ask yourself: is there a simpler or more general fact that would imply the statement you are trying to prove? Often, a stronger but simpler statement is actually easier to show than a more complicated one.

   If you think you've made progress, ask yourself: what additional detail would be enough to finish the problem?
5. **Don't give up.** It is rare that the very first thing you try to solve a problem will work in the first five minutes. Keep working, and remember that if something you tried didn't work, it still helped you understand the problem better.

It is often true that problems have many solutions; sometimes, one solution is much more elegant than another. I have seen this happen both at the Putnam level and in research papers—it happens that a ten-page proof has a follow-up that proves the same result in one paragraph. If you feel like you're getting bogged down in difficult calculations, this might mean that you're missing a simpler approach. On the other hand, it is more important to find a solution than to find the most beautiful solution; sometimes the right thing to do is to keep going, even when your approach looks messy!

## 3 Information

The webpage `https://facultyweb.kennesaw.edu/mlavrov/putnam.php` has details about all the Putnam-related events that will happen this semester.

It includes a Google form where you can sign up to receive emails about the Putnam competition and about Putnam practices. If you haven't already, I encourage you to fill it out!

The next Putnam practice is scheduled for **Saturday, September 28$^{\textbf{th}}$** in the same location: room 217 in the Atrium building.