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The two most important facts in graph theory

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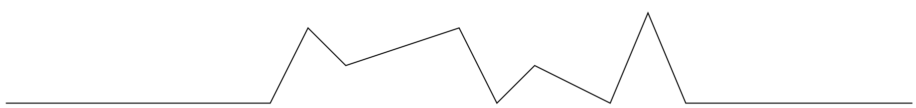
1 The handshake lemma

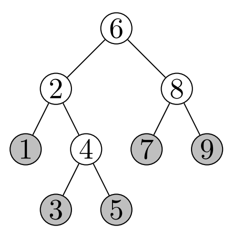
A **graph** is a structure for tracking some kind of relationship between pairs of objects. For example, you might have a bunch of guests at a party, and track which guests shook hands. The objects (e.g. the guests) are called **vertices**, the relationships (e.g. the handshakes) are called **edges**, and the number of edges a vertex is involved in (e.g. the number of times a guest shakes hands) is called its **degree**.

The handshake lemma says that **the sum of degrees in a graph is equal to twice the number of edges**. (In particular, it must always be even, which helps rule out some impossibilities!)

1. Every pair of people at a party is either friends or not friends. Prove that there are two people with the same number of friends.
2. I draw a maze with an entrance, an exit and no dead ends. Every intersection in the maze is a 4-way intersection. Prove that my maze is guaranteed to have a solution.
3. (Application of the maze problem) Two hikers start at ground level, at opposite sides of a 1D mountain range with piecewise linear slopes (an example is shown below). Prove that it's possible for them to meet on the mountain range while always walking in such a way that they remain at the same height.



4. In a full binary tree, every vertex is either a leaf or a parent of two children, as shown below:



   Prove that the number of leaves is always half of the total number of vertices, rounded up.

5. An open problem called *Conway's* 99*-graph problem* is to determine whether there is a 99-vertex graph with the following properties:
   - Every two adjacent vertices have exactly one common neighbor;[1]
   - Every two non-adjacent vertices have exactly two common neighbors.

   If such a graph exists, then every vertex in it must have the same degree, d. What is d?

[1] The neighbors of a vertex are the vertices it has an edge to; when there is an edge between two vertices, we say they are adjacent.

2 Hall's matching theorem

Suppose you have sets $A_1, A_2, \ldots, A_n$ and you want to pick elements $x_1 \in A_1, \ldots, x_n \in A_n$ such that all n elements $x_1, \ldots, x_n$ chosen are different.

If there are some k of your sets that together contain at most $k-1$ distinct elements, then this is definitely not possible! No matter how you choose x's from those k sets, there will be only $k-1$ distinct values among the k of them.

Hall's theorem says that this is the only obstacle: if you cannot find k such sets for any k, then it is possible to choose $x_1, \ldots, x_n$.

6. (Classic; you don't really need Hall) Two opposite squares are removed from an 8×8 chessboard. Can the remaining 62 squares be covered by 31 dominoes?

7. Alice draws a picture on a 8.5" by 11" sheet of paper that separates the sheet into 100 regions of equal area (and arbitrary shapes). Then, she hands the sheet of paper to Bob, who flips it over and does the same thing (but Bob's regions may be different shapes from Alice's).

   Prove that it's possible to make 100 holes in the paper such that each region on each side has a single hole through it.

8. King Kong and his gorilla friends are sharing some bananas. The gorillas are very picky and each is only interested in some of the bananas, but it's a rule that for every k gorillas, there are at least $2k$ bananas that could interest some of them. Prove that it's possible to give each gorilla *two* bananas that they like.

9. (Application of the King Kong problem) Prove that for large enough n, it's possible for the second player to force a draw in $n \times n \times n$ three-dimensional tic-tac-toe (where to win, you have to claim n cells in a line).

10. A **chain** of subsets of $\{1, 2, \ldots, n\}$ is a sequence $A_1 \subset A_2 \subset \cdots \subset A_k \subseteq \{1, 2, \ldots, n\}$. Prove that it's possible to decompose the power set of $\{1, 2, \ldots, n\}$ into $\binom{n}{\lfloor n/2 \rfloor}$ disjoint chains.

11. **Putnam 2012/B3.** A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

3 Information

The webpage `https://facultyweb.kennesaw.edu/mlavrov/putnam.php` has details about all the Putnam-related events that will happen this semester.

It includes a Google form where you can sign up to receive emails about the Putnam competition and about Putnam practices. If you are interested in either of these things, I encourage you to fill it out!

The next Putnam practice is scheduled for **Saturday, October 12$^{\text{th}}$** in room 217 in the Atrium building.