Mikhail Lavrov misha.p.l@gmail.com

# Putnam practice #2: Polynomials

*September 28, 2024* *Kennesaw State University*

## 1 Warm-up

A rectangle with area 18 and perimeter 20 is inscribed in a circle. What is the diameter of the circle?

## 2 Useful facts

**Fundamental theorem of algebra.** A degree-$n$ polynomial has exactly $n$ roots over the complex numbers, counted with multiplicity. If the roots are $r_1, \dots, r_n$, then the polynomial can be written as

$$a(x - r_1)(x - r_2) \cdots (x - r_n).$$

A useful consequence of this is that if two degree-$n$ polynomials agree in $n + 1$ points, then they must be equal. (Look at their difference!)

**Vieta's formulas.** Let $r_1, r_2, \dots, r_n$ be the roots (with multiplicity) of a degree-$n$ polynomial equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0.$$

Then the sum of the roots is $-a_{n-1}$, the product of the roots is $(-1)^n a_0$, and in general $(-1)^k a_{n-k}$ is the sum of all $k$-fold products of the roots. (For example, if $n = 4$, then $a_2$ gives us the sum of pairwise products: $r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4$.)

Sometimes this is stated with a coefficient $a_n$ on $x^n$, in which case the formulas apply to $(-1)^k \frac{a_{n-k}}{a_n}$.

**Rational root theorem.** If the polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_1, \dots, a_n$ are *integers* has a rational root $r = \frac{p}{q}$, then $p$ is a factor of $a_0$ and $q$ is a factor of $a_n$.

This is useful for finding nice roots of a specific equation.

## 3 Problems

1. Prove that if three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ on the curve $y = x^3 - x$ are collinear, then $x_1 + x_2 + x_3 = 0$. Explain how the "cubic curve calculator" on the last page works.
2. Suppose that a degree-2023 polynomial $P(x)$ satisfies $P(n) = \frac{1}{n}$ for $n = 1, 2, \dots, 2024$. Find $P(2025)$.
3. (AIME 2001) Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.

4. Suppose that the cubic curve $y = x^3 - 20x + 24$ intersects a circle centered at $(7, 24)$ in six distinct points $(x_1, y_1), \ldots, (x_6, y_6)$. Determine the sums $x_1 + \cdots + x_6$ and $y_1 + \cdots + y_6$.

5. (Putnam 1999) Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all $x$,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

6. (Putnam 1999) Let $P(x)$ be a polynomial of degree $n$ such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have $n$ distinct roots.

7. (Putnam 2004) Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$. Show that the $n$ numbers

$$\begin{aligned} &c_n r \\ &c_n r^2 + c_{n-1} r \\ &c_n r^3 + c_{n-1} r^2 + c_{n-2} r \\ &\cdots \\ &c_n r^{n-1} + c_{n-1} r^{n-2} + c_{n-2} r^{n-3} + \cdots + c_1 r \end{aligned}$$

are integers.

8. (Putnam 2005) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$.

## 4 Information

The webpage `https://facultyweb.kennesaw.edu/mlavrov/putnam.php` has details about all the Putnam-related events that will happen this semester.

It includes a Google form where you can sign up to receive emails about the Putnam competition and about Putnam practices. If you haven't already, I encourage you to fill it out!

The next Putnam practice is scheduled for **Saturday, October 12th** in the same location: room 217 in the Atrium building.

## 5 Cubic curve calculator

Take a ruler (or other straight object) and align it to two numbers $a$ and $b$ on the curve. Depending on how you choose the numbers, the third place where the ruler hits the curve will either be the product $ab$ or one of the quotients $a/b$ or $b/a$.