Mikhail Lavrov misha.p.l@gmail.com

Putnam practice #3: Number theory

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1 Warm-up

How many divisors does 10 000 have?

What is the sum of those divisors?

2 Problems

1. (AIME 1998) If a random divisor of 10^{99} is chosen, what is the probability that it is a multiple of 10^{88}?

2. For any three-digit number n, let $f(n)$ be the six-digit number obtained by writing two copies of n side by side. (For example, if $n = 125$, then $f(n) = 125125$.)

   Prove that for all three-digit numbers n, $f(n)$ is always divisible by 7. Find two other small primes that always divide $f(n)$.

3. (Putnam 1967) A locker room has lockers numbered 1 through n. On the first day, an attendant unlocks all the lockers. On the k^{th} day for $2 \le k \le n$, the attendant changes the state (locked to unlocked or vice versa) of all lockers whose numbers are divisible by k. After n days, which lockers are unlocked?

4. (Putnam 1969) Let n be a positive integer such that $n+1$ is divisible by 24. (We write: $24 \mid (n+1)$.) Prove that the sum of divisors of n is also divisible by 24.

5. A number n is called a "perfect number" if the sum of the proper divisors of n is n itself. For example, 28 is a perfect number, because $1 + 2 + 4 + 7 + 14 = 28$.

   Suppose that for some integer exponent $p > 1$, $2^p - 1$ is a prime number. Prove that the number $n = 2^{p-1}(2^p - 1)$ is a perfect number.

6. (Putnam 1956) Prove that every positive integer has a multiple whose decimal representation contains all 10 digits.

7. Prove that if $a \mid b$, then $\underbrace{111\dots111}_{a \text{ ones}} \mid \underbrace{111\dots111}_{b \text{ ones}}$.

8. (Putnam 1989) How many numbers in the sequence $101, 10101, 1010101, \dots$ are prime?

9. Find the number of zeroes at the end of the number $100! = 100 \cdot 99 \cdot 98 \cdots 3 \cdot 2 \cdot 1$.

10. (Putnam 1981) Let $E(n)$ denote the largest k such that the product $1^1 2^2 3^3 \cdots n^n$ is divisible by 5^k. Find the limit
$$\lim_{n\to\infty} \frac{E(n)}{n^2}.$$

11. (Chicken McNugget Theorem—a classic) Suppose that McDonalds decides to start selling Chicken McNuggets in two sizes: 5 pieces and 13 pieces.
    (a) Prove that it is impossible to purchase exactly 47 pieces through a combination of orders, and that 47 is the largest such integer.
    (b) Prove that for all positive integers $k < 47$, either you can purchase k pieces through a combination of orders, or you can purchase $47 - k$ pieces, but not both.
    (c) Prove that when the two sizes have a pieces and b pieces, the value of 47 is replaced by $ab - a - b$.

    (Misha's note: this is one of those cases where a Putnam question—in this case, the next one on this handout—is much easier if you're familiar with a classic problem. The Chicken McNugget theorem is not easy to prove on your own. Try to get as far as you can, but no matter how far you get, also try to see if you can use the answer to solve the next problem on this handout!)

12. (Putnam 1971) In a solitaire game, the player starts with 0 points and gets either a or b points every turn. There are 35 positive integers which can never be the player's score; one of them is 58. What are a and b?

3 Information

The webpage `https://facultyweb.kennesaw.edu/mlavrov/putnam.php` has details about all the Putnam-related events that will happen this semester.

It includes a Google form where you can sign up to receive emails about the Putnam competition and about Putnam practices. If you haven't already, I encourage you to fill it out!

The next Putnam practice is scheduled for **Saturday, October 26$^{\text{th}}$** in the same location: room 217 in the Atrium building.