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# Putnam practice #4: Linear algebra

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## 1 Warm-up

An $n \times n$ matrix $A$ is **symmetric** if $a_{ij} = a_{ji}$ for all $i$ and $j$, and **skew-symmetric** if $a_{ij} = -a_{ji}$ for all $i$ and $j$. Prove that every matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Try it on an example first:

$$\begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 3 \\ 6 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} + \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Then figure out what you did to solve the example and generalize.

## 2 Thinking in matrix multiplication

The rule for matrix multiplication is this: if $AB = C$, where $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix, then $C$ is an $m \times p$ matrix where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

But that is only part of the story…

1. (VTRMC 1987) Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices such that $A$ is invertible. Define $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ by $a_{ij}(t) = a_{ij}$ for $i < n$, $a_{nj}(t) = ta_{nj}$, $b_{ij}(t) = b_{ij}$ for $i < n$, and $b_{nj}(t) = tb_{nj}$. For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A(t) = \begin{bmatrix} 1 & 2 \\ 3t & 4t \end{bmatrix}$.

   Prove that $A(t)^{-1}B(t) = A^{-1}B$ for $t > 0$ and any $n$.

2. (a) Suppose that $A(t)B(t) = C(t)$, where $A(t)$ is an $m \times n$ matrix whose entries are functions of a real variable $t$, and $B(t)$ is similarly an $n \times p$ matrix function of the same type.

   Prove the product rule for matrix multiplication: $C'(t) = A'(t)B(t) + A(t)B'(t)$.

   (b) If $A(t)$ is an $n \times n$ matrix, and invertible for all $t$, find a rule for the derivative of $A(t)^{-1}$.

3. Show that

$$\det \begin{bmatrix} a_1 + b_1 x & a_1 x + b_1 & c_1 \\ a_2 + b_2 x & a_2 x + b_2 & c_2 \\ a_3 + b_3 x & a_3 x + b_3 & c_3 \end{bmatrix} = (1 - x^2) \cdot \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

## 3 When is the determinant 0?

The most important thing to know about the determinant of a matrix is to know whether it is 0 or not.

This indicates that there's some nontrivial linear relation between the rows of a matrix (and between the columns). It means that the matrix is not invertible. In geometric applications, it can mean that the matrix transforms the plane into a line or point (or transforms space into a plane, line, or point).

If a row or column of the matrix is 0, or if two rows or columns are equal or multiples of each other, that's a good way to see that the determinant is 0. Another way is to spot a small set of vectors such that all the rows (or all the columns) are linear combinations of those vectors.

4. (Putnam 2008) Alan and Barbara take turns filling in entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn, a player places a real number in a vacant entry. When all entries are filled, Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. (VTRMC 1982) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^3$ such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly dependent. Show that
$$\det \begin{bmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{bmatrix} = 0.$$
See if you can generalize to vectors in $\mathbb{R}^n$.

6. (Putnam 2009) Let $d_n$ be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. For example,
$$d_3 = \det \begin{bmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{bmatrix}.$$
(The argument of cos is always in radians, not degrees.)

   Evaluate $\lim_{n\to\infty} d_n$.

7. (Putnam 2003) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that, for all $x$ and $y$,
$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

## 4 Information

The webpage `https://facultyweb.kennesaw.edu/mlavrov/putnam.php` has details about all the Putnam-related events that will happen this semester.

It includes a Google form where you can sign up to receive emails about the Putnam competition and about Putnam practices. If you haven't already, I encourage you to fill it out!

The next Putnam practice is scheduled for **Saturday, November 9$^{\text{th}}$** in the same location: room 217 in the Atrium building.