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Putnam practice #5: Recurrence relations

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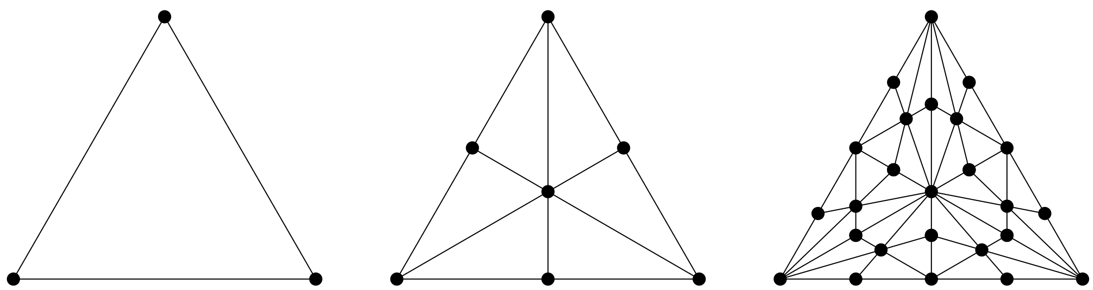
1 Warm-up

To create the **barycentric subdivision** of a triangle, we draw lines from each corner to the midpoint of the opposite side. The first two diagrams below show a triangle and its barycentric subdivision:



The barycentric subdivision of a triangle contains six smaller triangles, and so we can continue by taking the barycentric subdivision of each of them (as shown in the third diagram). This can be repeated.

I once became curious about the number of corner points in this iterated barycentric subdivision. If you count, then you see 3 in the original triangle, 7 in its barycentric subdivision, and 25 in the subdivision of the subdivision. I went one step further and counted 121 corner points in the next shape...

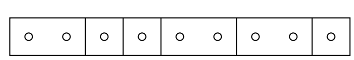
...at which point, I boldly conjectured that the n^{th} diagram in the sequence would have $(n+1)!+1$ corner points. After all, $2!+1=3$, $3!+1=7$, $4!+1=25$, and $5!+1=121$.

Prove my conjecture wrong. As a bonus, can you find a correct formula? (There is a way to prove the conjecture wrong without finding a correct formula, and without directly counting points in any diagram.)

2 Problems

1. The **Fibonacci sequence** is defined by the rule $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, with $F_1 = 1$ and $F_0 = 0$. The answer to each of the problems below, except for one is "the n^{th} Fibonacci number, F_n". Which one is the odd one out?

   (a) How many ways are there to tile a $1 \times (n-1)$ rectangle with 1×1 and 1×2 tiles? An example for $n = 10$ is shown below:

(b) How many ways are there to write n as an *ordered* sum of odd integers? (For example, we could write 10 as $1+5+3+1$, or as $3+1+1+5$, or as $5+5$, or in other ways.)

(c) How many ways are there to write $2n-3$ as an *unordered* sum of odd integers? (For example, we could write $5 = 2 \cdot 4 - 3$ as 5 or $3+1+1$ or $1+1+1+1+1$, but $1+3+1$ is considered the same as $3+1+1$.)

(d) How many integers k with $2^{n-1} \le k \le 2^n - 1$ have a binary representation in which no bit repeats three times in a row? (For example, when $n = 6$, then $51 = 110011_2$ counts, but $57 = 111001_2$ does not.)

2. (VTRMC 2006) Find the last digit of F_{2006}, the 2006^{th} Fibonacci number.

3. (AIME 2001) A mail carrier delivers mail to 19 houses in a row. No two adjacent houses ever get mail on the same day, and there are never more than three consecutive houses that get no mail on the same day. How many different patterns of mail delivery are possible?

4. If $a_0 = 4$ and $a_{n+1} = a_n^2 - 2$ for all $n \ge 1$, prove that there are integers p and q such that $a_n = (p+\sqrt{q})^{2^n} + (p-\sqrt{q})^{2^n}$.

   (This recurrence relation is used when testing large Mersenne numbers for primality.)

5. (Putnam 1985) Let d be a real number. For each integer $m \ge 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$\begin{aligned} a_m(0) &= d/2^m \\ a_m(j+1) &= (a_m(j))^2 + 2a_m(j), \qquad j \ge 0. \end{aligned}$$

   Evaluate $\lim_{n\to\infty} a_n(n)$.

6. (Putnam 1991) Let $S(n)$ be the difference between n and the largest perfect square less than n; for example, $S(11) = 11 - 3^2 = 2$.

   A sequence $a_0, a_1, a_2, \ldots$ is defined by the rule $a_{n+1} = a_n + S(a_n)$. For which starting values a_0 does this sequence eventually become constant?

3 Information

The webpage `https://facultyweb.kennesaw.edu/mlavrov/putnam.php` has details about all the Putnam-related events that will happen this semester.

It includes a Google form where you can sign up to receive emails about the Putnam competition and about Putnam practices. If you haven't already, I encourage you to fill it out!

Our last Putnam practice is scheduled for **Saturday, November 16th** in the same location: room 217 in the Atrium building. This will be an 90-minutes-long mini-Putnam session. I encourage everyone to attend who is considering taking the actual Putnam (though you're also welcome if you don't plan to do so).

4 Footnote

Suppose a sequence $x_1, x_2, \ldots$ satisfies the rule $x_n = ax_{n-1} + bx_{n-2}$ for some *constants* a and b.

Solve the quadratic equation $x^2 = ax + b$ for two roots $x = r_1, r_2$. Then there is a non-recurrent formula for x_n: it is $x_n = C_1 r_1^n + C_2 r_2^n$ (for some constants C_1, C_2 you'll have to figure out some other way, such as by using the initial conditions).

If the quadratic equation has a double root r, then the formula is $x_n = (C_1 + C_2 n) r^n$, instead.

All this generalizes, but this much is already fine to be going on with. This fact is occasionally very useful to finish the solution to a Putnam problem. (It usually does not need to be cited, because once you suspect that a formula for the sequence is true, you can prove it by induction.)