Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 10: Bridges and Trees

September 16, 2021 *Kennesaw State University*

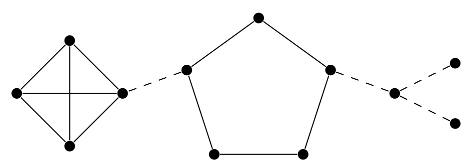
1 Bridges

We have already talked about connected components in graphs, and some example situations in which we care whether a vertex can be reached from another vertex.

The next step is to talk about how resilient graphs are. If a graph is connected, how hard is it to disconnect? If a graph has k connected components, how much do we need to do to make it $k+1$?

Chapter 5 of the textbook, which we'll get to *after* the first exam, deals with this question very generally. Right now, we'll just look at one special case: disconnecting graphs by deleting an edge.

We say that a **bridge** in a graph G is an edge $e \in E(G)$ such that $G - e$ has more components than G. Most often, we will be looking at a connected graph G; in this case, a bridge in G is an edge $e \in E(G)$ such that $G - e$ is not connected. For example, in the graph below, the dashed edges are the bridges:



We might care about identifying the bridges in a graph when we:

- Want to know what happens when network (say, a physical railway network, or a computer network) is damaged. A bridge is a critical connection that would disrupt communications between two points. We can avoid bridges, but this often comes at the expense of increasing the number of edges, or increasing the diameter of the network; both of these are also bad.
- Are trying to solve a puzzle like the Towers of Hanoi puzzle or the three cups puzzle. A bridge, in such a case, is a critical step that must be done to get from one state to another, and cannot be avoided.

In the previous lecture we proved the theorem that if e is any edge of the cycle graph C_n, then $C_n - e$ is still connected. We can now rephrase this theorem; it says "C_n has no bridges".

More is true along these lines:

Theorem 1.1. *An edge in a graph is a bridge if and only if it is not part of any cycles.*

[1] This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

Proof. When proving an if-and-only-if statement, the ordinary (and more careful) thing to do is to write two proofs: one for the "if" direction, and one for the "only if" direction. The brave and reckless option is instead to write a single proof, in which every step is an if-and-only-if step. We'll see how this goes here.

Let uv be an edge of a graph G. The statements "uv is a bridge" and "uv is not part of any cycles" only care about the connected component containing u and v. So we may assume G is connected; otherwise, just look at that connected component, instead of all of G.

The edge uv is *not* a bridge if and only if $G - uv$ is connected. This is by definition of a bridge (in the case where G is connected).

$G - uv$ is connected if and only if there is a $u - v$ path in $G - uv$. One direction of this follows by definition: if $G - uv$ is connected, certainly there must be a $u - v$ path in it. But if there is a $u - v$ path in $G - uv$, we can actually use it to find an $x - y$ walk in $G - uv$ for any vertices x and y. Just take an $x - y$ walk in G (because we know G is connected); whenever it goes from u to v, splice in the $u - v$ path. So just finding a $u - v$ path shows that $G - v$ is connected.

There is a $u - v$ path in $G - uv$ if and only if there is a cycle in G containing uv. If $(u, x_1, x_2, \dots, x_{k-1}, v)$ is a $u - v$ path in $G - uv$, then $(u, x_1, x_2, \dots, x_{k-1}, v, u)$ is such a cycle. Conversely, if $(x_0, x_1, \dots, x_{i-1}, u, v, x_{i+2}, \dots, x_k, x_0)$ is a cycle in G containing edge uv, then $(u, x_{i-1}, x_{i-2}, \dots, x_1, x_0, x_k, x_{k-1}, \dots, x_{i+2}, v)$ is a $u - v$ path in $G - uv$.

(If the last step is not clear from the description, draw a picture!)

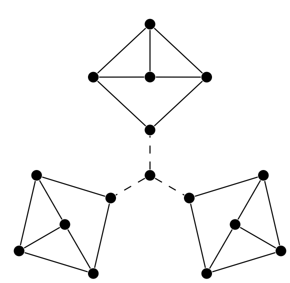
Chaining together the three bolded statements proves "The edge uv is not a bridge if and only if it is part of a cycle in G." This is equivalent to the statement "The edge uv is a bridge if and only if it is not part of any cycle in G", which was what we wanted. $\square$

The main use of this statement is as a shorter way to prove that a certain edge is not a bridge. But we'll see some interesting uses of it later on.

2 Bridges in regular graphs

In the previous lecture, we drew all the connected 3-regular graphs on 8 vertices. None of those graphs had bridges, because all of them contained an 8-cycle. This 8-cycle provides two ways to get from any vertex to any other (clockwise or counterclockwise around the cycle), and deleting any edge can only destroy one of those.

For larger n, however, there are many n-vertex 3-regular graphs that do have bridges. For example, below is a 3-regular graph with $n = 16$ vertices in which three edges are bridges:



We will see this particular graph reappear later in the semester, and the bridges it has will play a key role.

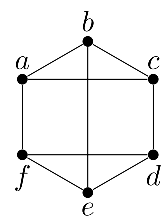
However, if you try to draw a 4-regular graph with a bridge, you will inevitably fail. There is a reason for this, which starts from the following lemma:

Lemma 2.1. *Let G be a graph; let $S \subseteq V(G)$; finally, let m_2 be the number of edges with 2 endpoints in S, and let m_1 be the number of edges with 1 endpoint in S. Then*

$$\sum_{v \in S} \deg(v) = 2m_2 + m_1.$$

Proof. This lemma generalizes the handshake lemma, and like that lemma, it can be proved by induction on the number of edges in G. However, there's a different approach that works both for the handshake lemma, and here.

Imagine listing some of the edges in G as follows: for each vertex $v \in S$, write down all the edges incident to v. For example, consider the graph below:



If $S = \{a, b, c\}$, we would write down the edges ab, ac, af then ab, bc, be then ac, bc, cd.

On the one hand, for every $v \in S$, we write down $\deg(v)$ edges. So the number of edges on our list is $\sum_{v \in S} \deg(v)$.

On the other hand, each edge with both endpoints in S is listed twice (once by each endpoint), and each edge with just one endpoint in S is listed once. So the number of edges on our list is $2m_2 + m_1$.

Therefore these two quantities are equal. $\square$

Corollary 2.2. *Let uv be a bridge in G, and let S be the set of vertices in the component of $G - uv$ containing u. Then $\sum_{v \in S} \deg(v)$ is odd.*

Proof. In this case, $G - uv$ has no edges from S to $V(G) - S$, because it's a connected component. So G has only one such edge: uv itself. Therefore in the previous lemma, $m_1 = 1$, and the value $2m_2 + m_1 = 2m_2 + 1$ is odd. $\square$

Corollary 2.3. *If all vertices of G have even degrees, then G has no bridges.*

Proof. If all vertices of G have even degrees, then $\sum_{v \in S} \deg(v)$ is a sum of even numbers, so it can never be odd for any S. □

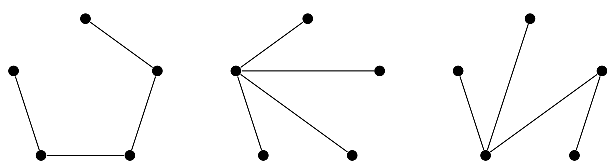
This last corollary applies, in particular, to 4-regular graphs.

3 Forests and trees

Two definitions:

- A **tree** is a connected graph in which all edges are bridges. In other words, if any edge of a tree is deleted, the tree stops being connected.
- A **forest** is any graph in which all edges are bridges. Every connected component of a forest must be a tree (which is where it gets its name).

Here are some pictures of trees—up to isomorphism, all the possible 5-vertex trees, in fact. You can also think of the entire diagram as a picture of one 15-vertex forest, if you prefer.



Although the definition of a forest seems simpler, trees are actually the more fundamental concept.

Trees are the simplest connected graphs. We will see later that every connected graph contains a **spanning tree**: a subgraph with the same set of vertices that is a tree. Because the spanning tree is itself connected, it's a short demonstration that we started with a connected graph.

One of the things that make trees very useful is that there are a million different alternative definitions.

Theorem 3.1. *The following are equivalent for a graph G (and so any of them makes G a tree):*

1. *G is minimally connected: it is connected, and deleting any edge destroys this.*
2. *G is connected and has no cycles.*
3. *G is maximally acyclic: it has no cycles, and adding any edge creates a cycle.*
4. *Between any two vertices of G, there is exactly one path.*
5. *G is connected and $|E(G)| \leq |V(G)| - 1$.*
6. *G is acyclic (has no cycles) and $|E(G)| \geq |V(G) - 1|$.*

Let's prove some of these equivalences: specifically, that 1, 2, and 3 are equivalent. (We'll look at some of the other ones next week.)

Lemma 3.2. *Condition 1 (our definition of a tree) is equivalent to condition 2.*

Proof. ($\Longrightarrow$) Suppose G is a tree; this means it's connected, so we're already halfway to condition 2. Then every edge is a bridge, so no edge is part of cycles. Therefore G can't have any cycles: there are no edges for those cycles to use.

($\Longleftarrow$) Now suppose G satisfies condition 2. Since it has no cycles, in particular for every edge e, there are no cycles containing e, which makes e a bridge. Therefore every edge of G is a bridge; also, condition 2 tells us that G is connected, so it is a tree. $\square$

At this point, we know a bit more about trees, which gives us more flexibility in proving the next part:

Lemma 3.3. *G is a tree if and only if it satisfies condition 3.*

Proof. ($\Longrightarrow$) Suppose G is a tree; by condition 2, it is acyclic. Let uv be any edge which is not already in G; we'll check that adding G creates a cycle.

$G + uv$ is also connected; moreover, deleting uv from $G + uv$ brings us back to G, so uv is not a bridge of G. Therefore uv must lie on some cycle in $G + uv$, and in particular $G + uv$ does have a cycle. Since uv was arbitrary, we conclude G is maximally acyclic: adding any edge creates a cycle.

($\Longleftarrow$) Suppose G is maximally acyclic. We'll prove it is also connected, which gives us condition 2.

Let u and v be any pair of vertices. If uv is an edge of G, then there is definitely a $u - v$ path: the path (u, v). Otherwise, we know that $G + uv$ has a cycle; in particular, a cycle containing uv, since there wasn't a cycle in G. Just as in the proof of Theorem 1.1, "$G + uv$ has a cycle containing uv" is equivalent to "G has a $u - v$ path". So either way, we find a $u - v$ path, concluding that G is connected. $\square$