Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 13: Cut vertices

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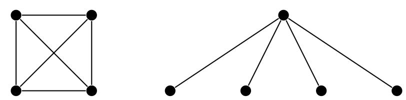
1 Plans for connectivity

Our goal for the next few lecture is to understand connectivity of graphs: if a graph is connected, how much has to be removed from it to disconnect it? Intuitively, a complete graph K_4 and a star graph $K_{1,4}$ are both connected, but the second is much more fragile: removing the top vertex will leave 4 isolated vertices.



The edge connectivity of a graph (which we'll define precisely later) looks at edge cuts: sets of edges that will disconnect the graph when removed. We have already seen the notion of a **bridge**: an edge whose removal increases the number of connected components. That's just an edge cut consisting of only one edge.

The vertex connectivity of a graph (again, we'll return to this later) looks at vertex cuts: sets of *vertices* that will disconnect the graph when removed. For now, we will just look at **cut vertices**: vertex cuts consisting of only one vertex.

One of the big problems here is that proving that a graph is "weak"—that you can remove a few pieces and it falls apart—is relatively easy. Simply show how to do it! On the other hand, proving that a graph is "strong"—that it remains connected if you don't delete too much from it—is hard, working from the definition. You would have to check that all the possible ways to remove something from the graph leave it connected.

("Weak" and "strong" are not technical terms here.)

For this reason, one of the big things we'll look for is alternative characterizations of graphs without a small edge or vertex cut. For example, we already know that an edge vw is a bridge if it is not part of any cycles. Conversely, if every edge of a graph G is part of some cycle, then we know that G has no bridges!

Today, we'll see a similar characterization of graphs with no cut vertices.

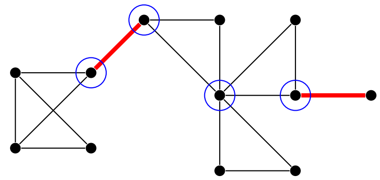
2 Cut vertices

Formally, we might say that a **cut vertex** in a graph G is a vertex v such that $G-v$ is disconnected. This definition is not very helpful if G is not connected, and in that case, we'd probably want to

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

say something like "$G - v$ has more connected components than G". But for the most part, we only care about cut vertices in connected graphs.

Consider the diagram below. The edges highlighted in red are bridges, which we're already familiar with. The vertices circled in blue are the cut vertices.



We see some similarities, but also some differences. Going from left to right in the diagram:

- Both endpoints of the leftmost bridge are cut vertices. Deleting an endpoint of an edge deletes that edge as well (and more), so this is a general trend. It's a trend with an exception, which we'll see in a moment.
- The third cut vertex from the left, however, is not part of any bridges. Deleting it leaves the graph in several pieces, but there is no single edge deletion that can accomplish the same thing.

  We also see here that cut vertices can disconnect a graph much more thoroughly than bridges. Removing an edge from a connected graph can leave at most 2 connected components, but removing a vertex can leave arbitrarily many connected components.
- With the second bridge, we see an exception to the trend: only one of its endpoints is a cut vertex. That's because deleting this bridge will isolate a vertex from the rest of the graph. Deleting that vertex will not disconnect the graph, because it destroys the entire piece that would end up being isolated!

The following claim is a slightly easier-to-work-with description of cut vertices. It does not add much to the definition, but it's a bit more convenient to use in proofs, because it's more specific.

Claim 2.1. *A vertex v in a connected graph G is a cut vertex if and only if there are two other vertices u, w in G such that all $u - w$ paths in G pass through v.*

Proof. If $G - v$ is not connected, there must be vertices u, w in different connected components of $G - v$. Because G is connected, there must exist one or more $u - w$ paths in G; however, because none of them survive in $G - v$, they must all pass through v.

On the other hand, if $G - v$ is connected, then for any two vertices u, w (not including v) there is a $u - w$ path in $G - v$. That's a $u - w$ path in G which does not pass through v. □

To see how this is useful to us, let's prove the observation from above about the relationship between bridges and cut vertices.

Corollary 2.2. *If vw is a bridge in G and $\deg(v) > 1$, then v is a cut vertex. But if $\deg(v) = 1$, then it's not.*

Proof. If $\deg(v) > 1$, then v has a neighbor u other than w. Can there be a $u - w$ path $(u, x_1, \dots, x_k, w)$ not passing through v? No: then $(u, x_1, \dots, x_k, w, v, u)$ would be a cycle containing vw, yet vw is a bridge and therefore not part of any cycles. Therefore u and w demonstrate that v is a cut vertex, by Claim 2.1.

On the other hand, if $\deg(v) = 1$, then there can be no such demonstration, and in fact v cannot be part of *any* path between two vertices that are not v. If the only neighbor of v is w, then any walk entering and leaving v has to do it via w: it must step $\dots, w, v, w, \dots$, and that's already not a path. □

So what happens in trees? In trees, every edge is a bridge. Therefore, every vertex that's not a leaf is a cut vertex. (We've already seen that if T is a tree and v is a leaf, then $T - v$ is still connected.)

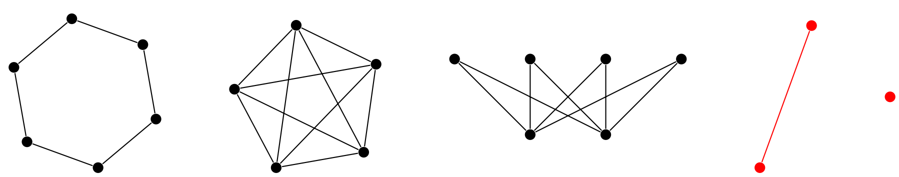
Every connected graph G has a spanning tree T, which has at least two leaves. For every leaf v of T, $G - v$ is still connected (because $T - v$ is a spanning tree of $G - v$) and so every graph has at least two vertices that are *not* cut vertices.

3 Nonseparable graphs

If a graph has a cut vertex, then this is easy to demonstrate: take the cut vertex, and show what happens when you remove it.

If an n-vertex graph has *no* cut vertices, then demonstrating this seems n times harder: for each of the n vertices, you show that the graph remains connected when you remove it. We'd like an easier strategy.

We'll call a connected graph with no cut vertices **nonseparable**.[2] Here are some examples of nonseparable graphs:



On the left, C_6 is one of the most straightforward examples: C_n is, in general, nonseparable. No matter which vertex you delete, the remainder is isomorphic to the connected graph P_{n-1}.

The next graph is K_5, which is nonseparable, but not for a very interesting reason. We already know that C_5 is nonseparable, and K_5 is just C_5 with some extra edges added. In general, adding more edges to a nonseparable graph can't stop it from being nonseparable: if $G - v$ is connected and you add more edges to G, $G - v$ will still be connected.

The next graph is $K_{4,2}$, which is also nonseparable. Delete a vertex from the top, and you'll get a graph isomorphic to $K_{3,2}$; delete a vertex from the bottom, and you'll get a graph isomorphic

[2] This is an *incredibly* similar definition to the definition of "2-connected", which is coming up in a future lecture. I will highlight the difference when we get there; watch out for this.

to $K_{4,1}$. This is a more interesting example, because $K_{4,2}$ does not contain a C_6 subgraph, so it's nonseparable for some other reason.

Finally, the two graphs in red are nonseparable, but in a weird way: they don't have enough vertices to have a cut vertex! (Looking back at Claim 2.1, for a vertex v to be a cut vertex, we need two other vertices u and w to exist, which means we need at least 3 vertices total.)

These last two graphs will often be a special case. In particular, we will exclude them from consideration in the following theorem, which characterizes nonseparable graphs.

Theorem 3.1. *A graph G on $n \geq 3$ vertices is nonseparable if and only if every two vertices of G lie on a common cycle.*

Proof of sufficiency. We will start with why this condition is sufficient: why it guarantees that G is nonseparable.

You should think of a cycle containing vertices v and w as giving us two completely separate $v - w$ paths. We get one $v - w$ path by going "clockwise" around the cycle, and another by going "counterclockwise". Except for v and w, these two paths have no vertices in common. (We will call two such paths "internally disjoint" later.)

If a vertex other than v and w is deleted from G, at least one of these paths still remains intact, because they don't share any vertices. So whenever we delete a vertex, there is still a $v-w$ path in the remainder (unless we deleted v or w). If *every* pair of vertices has this property, then *no* pair of vertices is disconnected when we delete a vertex. In particular, no cut vertex exists. □

Proof of necessity. Next, we will prove why this condition is necessary for a graph to be nonseparable. In other words, we will prove that if G is nonseparable with at least 3 vertices, then: for every two vertices v, w, there is a cycle in G containing both v and w.

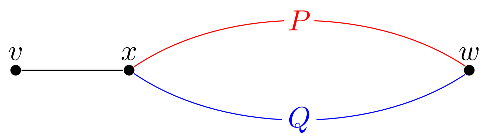
To do this, we induct on the distance $d(v, w)$. That is, we will begin by showing it for vertices v, w with $d(v, w) = 1$. Then, we'll show that if it's always true when $d(v, w) = k - 1$, it's also true when $d(v, w) = k$.

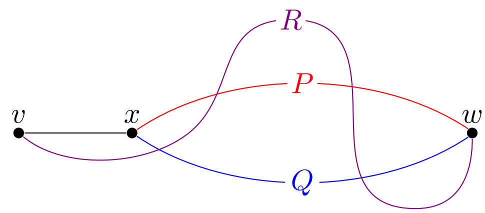
When $d(v, w) = 1$, v and w are adjacent. Also, we can't have $\deg(v) = \deg(w) = 1$, because then $\{v, w\}$ would be a connected component of G: they wouldn't have edges to any other vertices. This contradicts our assumption that G is a nonseparable (and therefore connected) graph on $n \geq 3$ vertices.

Suppose $\deg(v) > 1$. Then by a previous claim, if vw were a bridge, v would be a cut vertex. Since G is nonseparable, v cannot be a cut vertex, so vw cannot be a bridge. Finally, we know that an edge is a bridge if and only if it does not lie on any cycles; so there must be a cycle using edge vw. In particular, this is a cycle containing vertices v and w. So we have proven the base case of our result: the $d(v, w) = 1$ case.

Now assume that any two vertices at distance $k - 1$ lie on a common cycle, and let v, w be two vertices with $d(v, w) = k$. How do we use the induction hypothesis? Well, let x be the first vertex on some shortest path from v to w. This means $d(x, w) = k - 1$: to get from v to w in k steps, we first go to x, and then take $k - 1$ more steps to get from x to w.

By induction, x and w lie on a common cycle. As before, we interpret this as two $x - w$ paths that share no vertices other than x and w. We'll call these paths P and Q, giving us the first diagram below:





(Note that all of these diagrams represent only *part* of what's going on in the graph G: the part we've been able to prove exists!)

Actually, the diagram could look slightly different. It's possible that one of P or Q passes through v. In that case, we're already done: together, P and Q would form a cycle containing both v and w. So we will assume that v does *not* lie on P or Q: the diagram above is accurate.

We know x is not a cut vertex, because we know that G has no cut vertices. Therefore $G - x$ is still connected; in particular, there is still a $v - w$ path in $G - x$. Call such a path R.

We could end up in a nice and easy case where R does not share any vertices with P or Q. In that case, we can find a cycle that contains v and w: go from v to x, follow P (for example) to w, then follow R back to v. However, in general, R could share many points with both P and Q; we have a situation like the one shown in the second diagram above.

Without loss of generality, suppose that when we follow R from v to w, we intersect Q *before* we intersect P, as in the diagram. (Why is this without loss of generality? Because we currently don't know anything to distinguish P from Q. So if things were the other way around, we could just switch which $x - w$ path we're calling P, and which one we're calling Q.)

Now we can find a cycle containing v and w as follows. Begin by following the path R from v until we first get to a vertex of Q. From there, follow the remainder of Q until we get to w. We still have not touched any vertex of P, so we can follow P back to x. Finally, take the edge xv back to v, getting a cycle.

This proves that any two vertices v, w with $d(v, w) = k$ lie on a cycle, assuming this for vertices at distance $k - 1$. By induction on the distance, we conclude that this is true for any pair of vertices, proving the theorem. $\square$