Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 14: Blocks

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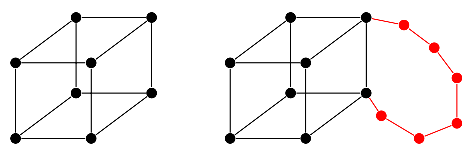
1 A digression on ears

Last time, we proved a tricky but powerful result about nonseparable graphs on at least three vertices: they are exactly the graphs in which every two vertices lie on a common cycle.

We will begin today's lecture by proving a few other properties of nonseparable graphs. These could also be turned into characterizations of nonseparable graphs, but we will not do that, because it takes time and effort.

An **ear** of a graph G is a path in G in which every vertex except the first and the last (every **internal** vertex) has degree 2. When we **add an ear** to a graph G, we pick two vertices v, w of G and create an ear by adding entirely new edges and (if the ear has length 2 or more) entirely new internal vertices to form a $v - w$ path. This is actually easier to show than to explain. Here is a cube graph, and a cube graph with an ear added:



Lemma 1.1. *If G is a nonseparable graph and we add an ear to G, the resulting graph is also nonseparable.*

Proof. Let H be a graph obtained from G by adding a $v - w$ ear with $k \geq 0$ internal vertices $x_1, x_2, \ldots, x_k$. (We will check that the new graph H still does not have a cut vertex.

To do this, we see what happens when we delete a vertex of H:

- Suppose we delete a vertex $u \in V(G)$, $u \neq v, w$. Because $G - u$ is connected, all vertices of $G - u$ are in the same connected component of $H - u$. Also, $x_1, x_2, \ldots, x_k$ all have a path to v and to w, so they are also in that same connected component: $H - u$ is connected.
- If we delete v or w, essentially the same thing happens. The only change is that for $x_1, x_2, \ldots, x_k$, we should observe that we still have a path to whichever of v or w we didn't delete.
- If we delete one of the new vertices x_i, then G remains connected (we didn't touch it); vertices $x_1, \ldots, x_{i-1}$ still have a path to v; vertices $x_{i+1}, \ldots, x_k$ still have a path to w. As a result, $H - x_i$ is still connected.

In all cases, H cannot be disconnected by deleting a single vertex, so H is nonseparable. □

[1] This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

This lemma gives us another way to demonstrate that a graph G is nonseparable without doing lots of work. Start with a very small nonseparable graph and keep adding ears to it until you get G. By the lemma, we have a nonseparable graph at every step, and in particular G is nonseparable. This is called finding an **ear decomposition** of G.

(At least, it does not require lots of work to write down the ear decomposition. Finding one may be slightly trickier!)

A theorem of Whitney from 1932 says that every nonseparable graph has an ear decomposition, so this strategy always works. We will not prove this, because the lemma above is enough for us today.

Corollary 1.2. *If G is a nonseparable graph and u_1v_1, u_2v_2 are any two different edges of G, then there is a cycle through both edges.*

Proof. To be clear: we do not need to assume that all four vertices u_1, v_1, u_2, v_2 are different. The proof we'll see may appear to assume this, but the same argument works if one of the vertices is repeated.

Add two ears to G: the ear (u_1, x, v_1) and the ear (u_2, y, v_2), where x and y are two completely new vertices. By Lemma 1.1, the resulting graph H is still nonseparable.

By what we proved last time, there is a cycle C in H containing both x and y. It might look, for example, like

$$C = (\dots, u_1, x, v_1, \dots, u_2, y, v_2, \dots)$$

though u_1, v_1 and u_2, v_2 do not need to appear in that order. If one of u_1, v_1 is equal to one of u_2, v_2, then the two segments of C containing x and y might even appear right next to each other.

Take the cycle C and remove x and y from it, getting something like:

$$(\dots, u_1, v_1, \dots, u_2, v_2, \dots).$$

This is still a cycle, because u_1 is adjacent to v_1 and u_2 is adjacent to v_2. Also, because it does not involve x or y, it is a cycle in G. In fact, it's the cycle we wanted to find: it traverses both the edge u_1v_1 and the edge u_2v_2. □

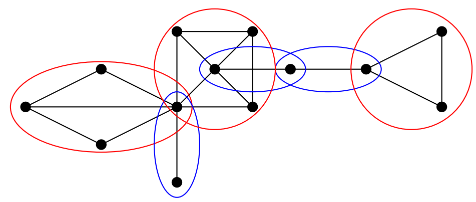
Once again, much more than what we've shown is true. The converse of this corollary also holds: if G is a connected graph, and every pair of edges in G lies on a cycle, then G is nonseparable. We will not prove this.

2 Block decomposition

When graphs are not connected, we split them up into connected components. Many problems can be solved by considering each connected component separately.

When graphs are not nonseparable, we can split them up into nonseparable pieces called blocks. Many problems can be solved by considering each block separately.

What exactly should blocks be? One way to define a block is to say "it's a nonseparable subgraph that isn't part of any larger nonseparable subgraph". Under this definition, let's look at the blocks of one example graph:



Let's make some observations about this graph, and then prove that they hold in general.

Claim 2.1. *Every bridge in G is a 2-vertex block, and all 2-vertex blocks are bridges.*

Proof. Every edge by itself is a nonseparable subgraph. So it's a question of whether it's part of a larger nonseparable subgraph: if it's not, then it must be a block by itself.

If an edge is *not* a bridge, then it is part of a cycle, and that cycle is a larger nonseparable subgraph. Therefore only the bridges can be 2-vertex blocks.

By Lemma 1.2, every block on more than 2 vertices has a cycle going through any pair of edges; in particular, every edge is part of a cycle. In particular, none of the edges of a larger block can be bridges. Therefore all bridges must be 2-vertex blocks, not part of larger blocks. □

Claim 2.2. *Two blocks of a graph G share at most one vertex.*

Proof. Suppose H_1, H_2 are nonseparable subgraphs of G, and they share at least two vertices. Then $H_1 \cup H_2$ is also nonseparable: if you delete any vertex v from $H_1 \cup H_2$, then

- any two vertices of H_1 are still joined by a path, because H_1 is nonseparable;
- any two vertices of H_2 are still joined by a path, becuase H_2 is nonseparable;
- there is at least one vertex left that's in both H_1 and H_2, and it has a path to all other vertices.

So H_1, H_2 are not blocks: they're part of a bigger nonseparable subgraph. Conversely, if they *are* blocks, then they can only share at most one vertex. □

Claim 2.3. *If two blocks H_1, H_2 of G overlap at a vertex v, then v is a cut vertex of G.*

Proof. Suppose v is not a cut vertex. Then if we pick a vertex $v_1 \neq v$ in H_1 and a vertex $v_2 \neq v$ in H_2, there must be a $v_1 - v_2$ path P that does not pass through v.

In that case, the union of H_1, H_2, and P is a bigger nonseparable graph, which cannot happen! If we delete a vertex from this union, then

- any two vertices of H_1 are still joined by a path, because H_1 is nonseparable;
- any two vertices of H_2 are still joined by a path, becuase H_2 is nonseparable;

- if a vertex from P was deleted, then H_1 and H_2 are connected through v, and each vertex of P has a path either to H_1 or to H_2;
- if no vertices of P are deleted, then it connects H_1 and H_2.

Therefore no such path P can exist, and so P is a cut vertex. □

Claim 2.4. *If vertices v, w are part of the same block, then every $v - w$ path in G is contained in that block.*

Proof. Let H be the block containing v and w. Any $v - w$ path either always stays in H (in which case we're happy) or it eventually leaves H; in that case, suppose it leaves H via a vertex x, and returns to H via a vertex y.

But then, adding the $x - y$ segment of that path to H is just like adding an ear to H; it makes a bigger nonseparable subgraph of G. This cannot happen; so the $v - w$ path must always stay in H. □

2.1 Another way to define blocks

There are two ways to define connected components of a graph. One of them is to do something like what we did above: say that they are the connected subgraphs that are not part of any larger connected subgraph. But the way we actually did it is through the equivalence classes of an equivalence relation.

This can be done with blocks, too. But an equivalence relation in the world of blocks can't possibly partition the *vertices* of the graph into equivalence classes—after all, we see that some vertices appear in multiple blocks! Instead, the equivalence relation partitions the *edges* of the graph.

Given a graph G, we define a relation $\sim$ on $E(G)$ with $e_1 \sim e_2$ if $e_1 = e_2$ or if there is a cycle that traverses both edges. This is automatically reflexive, and it doesn't take too much work to prove that it's symmetric. Proving that it's transitive is the hard part—but it is.

The equivalence classes of $\sim$ are sets of edges. It turns out that each equivalence class is the set of edges in one of the blocks of G. From Lemma 1.2, we can see that two edges in a block *must* be related by $\sim$. Conversely, if two edges are related by $\sim$, then the cycle containing them is a nonseparable subgraph—so that cycle (and those two edges) must be part of a single block.

In other words, two edges in different blocks can never be part of the same cycle!

3 Applications of blocks

Let's see a few places where we can use blocks to simplify thinking about a graph.

Proposition 3.1. *A graph G is bipartite if and only if every block of G is bipartite.*

Proof. To check if G is bipartite, we just need to know that it has no odd cycles. But a cycle cannot use edges from more than one block; so if G has an odd cycle, then that odd cycle is contained entirely in one block, and that block already is not bipartite. Therefore if all blocks are bipartite, then G is bipartite. □

This is a bit of a “toy” example: it is not actually that hard to check if a graph is bipartite. To take a more complicated example, let’s look at the number of spanning trees in G: one of the hardest things to calculate about graphs that we’ve encountered so far.

Proposition 3.2. *If G is connected and has blocks $H_1, H_2, \dots, H_k$, then $\tau(G)$, the number of spanning trees in G, satisfies*

$$\tau(G) = \tau(H_1) \cdot \tau(H_2) \cdots \tau(H_k).$$

Proof. The formula comes from the idea (which we’ll prove) that if we take a spanning tree of every block, their union is a spanning tree of G, and every spanning tree is obtained in this way.

The union of spanning trees of the blocks is acyclic: there is no cycle within a block (because within each block, we have a tree) and there is no cycle spanning multiple blocks (because there weren’t any such cycles in G to begin with). It is also connected: for any two vertices v, w you can take a $v - w$ path in G, and then replace the portion of that path within a block by a path within the spanning tree of that block. Therefore the union is a tree.

To prove that every spanning tree is obtained in this way: if we take a spanning tree T of G, then the piece of T within a block H_i is an acyclic subgraph (because T is acyclic). The piece of T within H_i is also connected: given vertices $v, w \in V(H_i)$, there is a $v - w$ path in T (because T is connected) which stays entirely in H_i (by Claim 2.4). □

There are other facts like this that we don’t have the tools to talk about. For example:

Claim 3.3. *A graph G is planar (it can be drawn in the plane without crossing edges) if and only if every block of G is planar.*

Claim 3.4. *A graph G is k-colorable if and only if every block of G is k-colorable.*

Claim 3.5. *The clique number of G (the number of vertices in the largest complete subgraph of G) is the maximum of the clique numbers of any of its blocks.*