Math 3322: Graph Theory[1] Mikhail Lavrov

# Lecture 16: Menger's Theorem

*October 12, 2021* *Kennesaw State University*

## 1 Menger's theorem

Menger's theorem states:

**Theorem 1.1.** *Let $s, t$ be two non-adjacent vertices of a graph $G$. Then we can find a set of $\kappa(s,t)$ paths from $s$ to $t$, in which any two paths are internally disjoint: they share no vertices other than $s$ and $t$.*

The motivation here is an idea from the previous lecture: if we find a set of $k$ internally disjoint[2] $s-t$ paths, this proves that $\kappa(s,t)$ is at least $k$, because in order to disconnect $s$ from $t$, we must delete at least one internal vertex from each path. Menger's theorem says that whenever $\kappa(s,t) = k$, this strategy always works to prove that $\kappa(s,t)$ is at least $k$.

This is very convenient, because the definition of $\kappa(s,t)$ is a minimization problem: we are finding the minimum number of vertices in an $s-t$ cut. This definition makes it easy to prove upper bounds on $\kappa(s,t)$ (just find an $s-t$ cut that you think is good) but hard to prove lower bounds (how do you check that there is no better $s-t$ cut?). Using internally disjoint paths is good for lower bounds, and when the two bounds meet in the middle, you know that you've found $\kappa(s,t)$.

### 1.1 The structure of the proof

Our proof of this theorem (following a 1966 proof of Gabriel Dirac) will use the "minimum counterexample" technique. Suppose, for the sake of contradiction, that Menger's theorem is false. Then choose the graph $G$ and vertices $s, t \in V(G)$ to be a case where the theorem fails such that $G$ has as few edges as possible. Next, we:

- Use the fact that Menger's theorem is true for all graphs with fewer edges than $G$ to deduce properties of $G$.
- Try to arrive at a contradiction, which will let us conclude that Menger's theorem is true.

In spirit, this is very similar to a proof by induction on the number of edges in $G$; in either case, we get to assume that Menger's theorem holds for graphs smaller than $G$. In fact, the textbook gives a version of this proof that *is* phrased as an induction proof; you can read that for a different take on the argument.

For the rest of this section, we will assume that $G$ is a specific minimal counterexample, in which $s, t$ are two non-adjacent vertices with $\kappa(s,t) = k$ but with no set of $k$ internally disjoint $s-t$

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

[2]Technically, I should be saying *pairwise* internally disjoint here, to emphasize that no two of the paths share an internal vertex. I will drop the "pairwise" in today's lecture notes to avoid the cumbersome language.

paths.

We will assume $k \geq 2$. That's because Menger's theorem doesn't say very much when $k = 0$ (in which case we have nothing to find) or when $k = 1$ (in which case, because $\kappa(s,t) > 0$, an $s-t$ path must exist, which is all we need).

**A note on notation.** You may have noticed that the notation $\kappa(s,t)$ does not specify the graph $G$ at any point, even though it definitely does depend on the graph $G$. So we will write $\kappa_G(s,t)$ when we want to specify the graph.

### 1.2 Deleting an edge from $G$

The easiest thing to do when using the minimality of $G$ is to see what happens when we delete an edge. We know that for any edge $xy \in E(G)$, Menger's theorem must hold for $G - xy$; otherwise, $G - xy$ would be a smaller counterexample.

First of all, here's a general claim about $s-t$ cuts in $G - xy$.

**Claim 1.2.** *Let $xy \in E(G)$ be any edge, and let $U$ be an $s-t$ cut in $G - xy$. Then:*

- *The set $U \cup \{x\}$ is an $s-t$ cut in $G$, unless $x = s$ or $x = t$.*
- *The set $U \cup \{y\}$ is an $s-t$ cut in $G$, unless $y = s$ or $y = t$.*

*Proof.* We know that $(G - xy) - U$ is a graph in which there is no way to get from $s$ to $t$; that's what it means for $U$ to be an $s-t$ cut in $G - xy$.

The graph $G - (U \cup \{x\})$ is a subgraph of $(G - xy) - U$: deleting $x$ in particular gets rid of edge $xy$. Therefore there is also no way to get from $s$ to $t$ in $G - (U \cup \{x\})$. This means that $U \cup \{x\}$ is an $s-t$ cut in $G$... except when $x = s$ or $x = t$, because an $s-t$ cut is not allowed to delete either of these vertices.

Similar logic applies to $U \cup \{y\}$. $\square$

By using the fact that $G$ is a minimal counterexample, we can say more.

**Claim 1.3.** *Let $xy \in E(G)$ be any edge, and let $H = G - xy$. Then $\kappa_H(s,t) = k - 1$.*

*Proof.* Without even using minimality, there are two options for what $\kappa_H(s,t)$ can be. We know that $\kappa_H(s,t) \leq k$, because there is a $k$-vertex $s-t$ cut in $G$, which remains a $k$-vertex $s-t$ cut in $H$. However, we can go the other way: if $U$ is an $s-t$ cut in $H$, then by Claim 1.2, either $U \cup \{x\}$ or $U \cup \{y\}$ is an $s-t$ cut in $G$. From $|U \cup \{x\}| \geq k$ we get $|U| \geq k - 1$.

Therefore $\kappa_H(s,t)$ is always either $k-1$ or $k$.

But because $G$ is a minimal counterexample to Menger's theorem, $\kappa_H(s,t)$ cannot be $k$. If it were, then by Menger's theorem applied to $H$, we would be able to find a set of $k$ internally disjoint $s-t$ paths in $H$. But they would also be a set of $k$ internally disjoint $s-t$ paths in $G$, which we assumed does not exist!

This leaves $\kappa_H(s,t) = k - 1$ as the only option. $\square$

## 1.3 Deleting a vertex from $G$

For every vertex $v$, we can reason similarly about the graph $H = G - v$. But we'll only do this in one specific case:

**Claim 1.4.** *$G$ does not contain any vertex $v$ adjacent to both $s$ and $t$.*

*Proof.* Suppose such a vertex $v$ exists. Let $H = G - v$.

If $U$ is an $s - t$ cut in $H$, then $U \cup \{v\}$ is an $s - t$ cut in $G$, because deleting $U \cup \{v\}$ from $G$ is the same as deleting $U$ from $H$. Therefore $|U \cup \{v\}| \geq k$, which means $|U| \geq k - 1$. Since this is true for all $U$, we have $\kappa_H(s, t) \geq k - 1$.

Because $H$ has fewer edges than $G$, we know that Menger's theorem holds for $H$; we can find a set of $k - 1$ internally disjoint $s - t$ paths in $H$. To that set of paths, add the $s - t$ path $(s, v, t)$: it is internally disjoint from all of them, because its only internal vertex is $v$, which does not exist in $H$. But this gives us a set of $k$ internally disjoint $s - t$ paths in $G$: the thing that Menger's theorem promises us! This would mean that $G$ is not actually a counterexample to Menger's theorem.

We conclude that no such vertex $v$ can exist. □

## 1.4 Two smaller problems

Let $U = \{u_1, u_2, \ldots, u_k\}$ be a particular $k$-vertex $s - t$ cut in $G$. Because $s$ is not connected to $t$ in $G - U$, we can split up $G$ into two pieces:

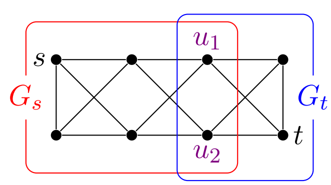
- $G_s$ is the induced subgraph "between $s$ and $U$". It includes all vertices that can be reached by a path from $s$ without going through a vertex in $U$ (but including the vertices in $U$ themselves).

  Because $U$ is an $s - t$ cut, $G_s$ does not contain the vertex $t$; it may be missing some other vertices.

- $G_t$ is the induced subgraph "between $U$ and $t$". It includes $U$, and all the vertices that *cannot* be reached by a path from $s$ without going through a vertex in $U$.

  The only vertices $G_s$ and $G_t$ have in common are the vertices of $U$.

Here's an illustration in the case $k = 2$:




We can use this decomposition to prove the following property of $G$ (which, remember, is the minimum counterexample to Menger's theorem).

**Claim 1.5.** *When splitting $G$ up into $G_s$ and $G_t$, either $G_s$ has only the $k+1$ vertices $\{s, u_1, u_2, \ldots, u_k\}$, or else $G_t$ has only the $k + 1$ vertices $\{u_1, u_2, \ldots, u_k, t\}$.*

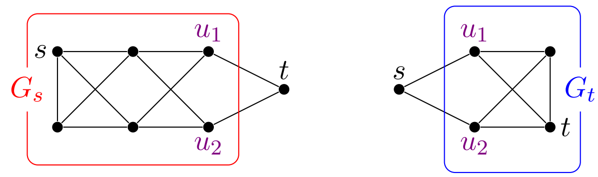
*In particular, if $U$ is any $k$-vertex $s-t$ cut in $G$, then either every vertex of $U$ is adjacent to $s$, or else every vertex of $U$ is adjacent to $t$.*

*Proof.* Suppose that both $G_s$ and $G_t$ have more than $k+1$ vertices. Then we can draw two smaller graphs than $G$.

In the first graph, $G'$, leave $G_s$ alone and replace $G_t$ by only the single vertex $t$ adjacent to $u_1, \ldots, u_k$.

In the second graph, $G''$, leave $G_t$ alone and replace $G_s$ by only the single vertex $s$ adjacent to $u_1, \ldots, u_k$.

Here's an illustration of $G'$ and $G''$ for our previous example:




Neither $G'$ nor $G''$ has a $(k-1)$-vertex $s-t$ cut: such an $s-t$ cut would also have to be an $s-t$ cut in $G$. Therefore $\kappa_{G'}(s,t) = \kappa_{G''}(s,t) = k$ (since $U$ is still an $s-t$ cut in both).

Both $G'$ and $G''$ have fewer edges than $G$, so Menger's theorem holds for both of them. So we can find $k$ internally disjoint $s-t$ paths in $G'$ and in $G''$.

But these can be glued together to get $k$ internally disjoint $s-t$ paths in $G$; here's how. For each $i$, take the $s-t$ path we found in $G'$ that goes through $u_i$: $(s, (\text{some stuff}), u_i, t)$. Also, take the $s-t$ path we found in $G''$ that goes through $u_i$: $(s, u_i, (\text{some other stuff}), t)$. Join them together at $u_i$:

$$(s, (\text{some stuff}), u_i, (\text{some other stuff}), t).$$

These are internally disjoint: they can't intersect in $G_s$ (because otherwise the corresponding paths in $G'$ would also intersect) and they can't intersect in $G_t$ (because otherwise the corresponding paths in $G''$ would also intersect).

This is a contradiction: $G$ was supposed to be a counterexample to Menger's theorem. The only way out, now, is if one of $G'$ or $G''$ was actually the same as $G$. In that case, it wouldn't actually have fewer edges than $G$, and we couldn't apply Menger's theorem to it. □

## 1.5 Completing the proof

Now we will combine all the claims we've found to get a contradiction.

Take a shortest $s-t$ path in $G$. This path has length at least 3: $s$ and $t$ are not adjacent, and by Claim 1.4, there are no paths of length 2. So it begins

$$(s, x, y, \ldots, t).$$

Note that $x$ is not adjacent to $t$ (by Claim 1.4) and $y$ is not adjacent to $s$ (because otherwise, we could get a shorter $s-t$ path by skipping $x$).

By Claim 1.3, $G - xy$ has an $s - t$ cut $U$ with $k - 1$ vertices. By Claim 1.2, both $U \cup \{x\}$ and $U \cup \{y\}$ are $k$-vertex $s - t$ cuts in $G$.

The cut $U \cup \{x\}$ contains vertex $x$, which is adjacent to $s$ but not to $t$. By Claim 1.5, since not all of $U \cup \{x\}$ is adjacent to $t$, all of $U \cup \{x\}$ must be adjacent to $s$. In particular, all of $U$ is adjacent to $s$. By Claim 1.4, no vertex can be adjacent to both $s$ and $t$, so in particular no vertex in $U$ is adjacent to $t$.

But now the cut $U \cup \{y\}$ contradicts Claim 1.5: none of the vertices in $U$ are adjacent to $t$, and $y$ is not adjacent to $s$.

Since we got a contradiction by taking a minimal counterexample to Menger's theorem, the theorem must hold for all graphs.

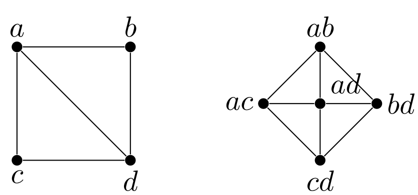
## 2 Line graphs and Menger's theorem for edge cuts

There is a version of Menger's theorem for edge connectivity:

**Theorem 2.1.** *Let $s, t$ be two vertices of a graph $G$. Then we can find a set of $\kappa'(s,t)$ paths from $s$ to $t$, in which any two paths are edge-disjoint: they share no edges.*

It would be a shame if we had to give another long proof of this theorem. Fortunately, there's a tool we can use to avoid starting from scratch.

The **line graph** $L(G)$ of a graph $G$ is a graph with a vertex for every edge of $G$. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ share an endpoint. Here is an example of a graph and its line graph:

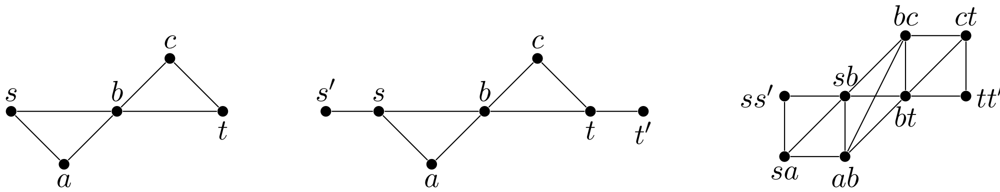



Lots of problems about edges turn into problems about vertices of the line graph, and this one is no exception. Here is how we prove the edge version of Menger's theorem from the vertex version:

*Proof of Theorem 2.1.* Starting from the graph $G$ and vertices $s, t$, do the following two steps (shown in an example on the next page):

1. Add a vertex $s'$ adjacent only to $s$, and a vertex $t'$ adjacent only to $t$; call the result $G'$
2. Take the line graph $L(G')$.

The graph $L(G')$ does not have vertices $s$ and $t$ anymore, but has a vertex $ss'$ and a vertex $tt'$. Moreover, whenever deleting some vertices of $L(G')$ disconnects $ss'$ from $tt'$, deleting the corresponding edges of $G$ will disconnect $s$ from $t$. Therefore if $\kappa'(s,t) = k$ in $G$, then $\kappa(ss', tt') \geq k$ in $L(G')$.




By Theorem 1.1, we can find $k$ $ss' - tt'$ paths in $L(G')$ that share no vertices other than $ss'$ and $tt'$. *(In the example, we can take the paths* $(ss', sb, bc, ct, tt')$ *ad* $(ss', sa, ab, bt, tt')$*.)*

Those paths turn into $s' - t'$ paths in $G'$ that share no edges other than $ss'$ and $tt'$. *(In the example, our paths turn into* $(s', s, b, c, t, t')$ *and* $(s', s, a, b, t, t')$*. Not all paths in* $L(G)$ *turn into paths in* $G$*, but they'll always turn into something that contains a path in* $G$*.)*

Finally, when we delete $s, t$ from all of these paths, they turn into $s - t$ paths in $G$ that share no edges at all. *(In the example, these will be* $(s, b, c, t)$ *and* $(s, a, b, t)$*.)* □