Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 20: Tournaments

October 26, 2021 *Kennesaw State University*

1 Definitions

A **tournament** is a special kind of directed graph. It has no loops, and for every pair of vertices v and w, exactly one of the possible arcs (v, w) or (v, w) exists.

You should imagine an actual (round-robin) tournament in which V is the set of contestants. If $v, w \in V$, then at some point v plays against w. If v beats w, then we record that information with the arc (v, w), and if w beats v, then we record that information with the arc (w, v). This is only one of the ways tournaments can come up in applications, but it's good for intuition.

In some ways, the world of tournaments is just as rich as the world of undirected graphs. If we fix a vertex set $V = \{v_1, v_2, \dots, v_n\}$, then there are $2^{\binom{n}{2}}$ possible graphs with that vertex set: for each of the $\binom{n}{2}$ pairs $\{v_i, v_j\}$, choose whether edge $v_i v_j$ is present or not. Similarly, there are $2^{\binom{n}{2}}$ possible tournaments with that vertex set: for each of the $\binom{n}{2}$ pairs $\{v_i, v_j\}$, choose whether arc (v_i, v_j) or arc (v_j, v_i) is present.

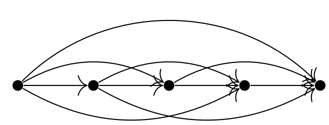
1.1 Transitive tournament

In the world of undirected graphs, there are two "extreme cases": the empty graph, and the complete graph. In the world of tournnaments, there is just one "extreme case" (or $n!$, depending on how you count). Here it is:

Theorem 1.1. *Up to isomorphism, there is only one acyclic tournament.*

Proof. We know that every acyclic digraph has a topological ordering: we can rename its vertices $v_1, v_2, \dots, v_n$ such that only arcs (v_i, v_j) with $i < j$ exist.

If we do this to an acyclic tournament, we know exactly which arcs exist: *all* the possible arcs (v_i, v_j) with $i < j$. For example, when $n = 5$:



So there is only one possible acyclic tournament we could get. This is "up to isomorphism", because the step where we renamed the vertices to respect the topological ordering is exactly an isomorphism of directed graphs. □

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

There are $n!$ possible ways to order n vertices, so if we don't say the magic words "up to isomorphism", there are $n!$ possible acyclic tournaments on a fixed vertex set of size n. For every possible ordering of the n vertices, direct all the arcs to go from the earlier vertex to the later vertex in the ordering.

An acyclic tournament is called a **transitive tournament**. If we interpret arc directions as comparisons, then a transitive tournament is one where all comparisons satisfy the transitive property: if there are arcs $u \to v$ and $v \to w$, there is also an arc $u \to w$. (This is analogous to "if $u > v$ and $v > w$, then $u > w$".)

This is not *obviously* equivalent to "there are no cycles". We could prove that the two properties are equivalent with a long argument now, or we could wait until later in this lecture when we'll be able to do it in one line.

1.2 Score sequences

When we discussed indegrees and outdegrees in class, the question was asked: do directed graphs have degree sequence?

Certainly it is possible to record a sequence of pairs $(\deg^+(v), \deg^-(v))$ for every vertex v, but this is a cumbersome and confusing object. There's no perfect way to sort it from largest to smallest, for example; how do you compare pairs like $(1, 5)$ and $(3, 2)$? We could peel apart the indegree and outdegree sequences, but then we lose the information of which goes with which.

With tournaments, the situation is much nicer. If the tournament has n vertices, then an outdegree of $\deg^+(v)$ always goes with an indegree of $\deg^-(v) = n - 1 - \deg^+(v)$, because the total number of arcs that start or end at v is $n - 1$. So we capture all the information there is by writing down the sequence of outdegrees.

This is also called the **score sequence** of the tournament. That's because, in the round-robin competition analogy, your outdegree is exactly your score: the number of times you've won against another participant.

Claim 1.2. *Up to isomorphism, there is only one tournament with score sequence* $0, 1, 2, \dots, n-1$*: the transitive tournament.*

Proof. Induct on n. When $n = 1$, the tournament with score sequence 0 is definitely a transitive tournament because there's nothing for it to be. Now assume this holds for $n - 1$ and let's try to prove it for n.

The vertex with outdegree $n - 1$ beats everyone else. Delete that vertex, and we get a tournament with score sequence $0, 1, 2, \dots, n - 2$. (Nobody else's score decreases, because we deleted a vertex that nobody else beat.) By induction, we're left with a transitive tournament on $n - 1$ vertices. When we add the vertex with outdegree $n - 1$ back in, we can put it first in the topological ordering; all arcs out of that vertex will go forward, so the topological ordering still works, and our tournament is still acyclic. Therefore it's transitive. $\square$

The score sequence of an n-vertex tournament has values between 0 and $n-1$, and the sum of all n scores should be $\binom{n}{2} = \frac{n(n-1)}{2}$: the total number of arcs in the tournament. But not all sequences with this property are score sequences. For example, consider the sequence

$$1, 1, 1, 1, 5, 6, 6$$

which does satisfy $1+1+1+1+5+6+6 = \binom{7}{2}$. What goes wrong here? Well, if we just look at the four vertices that have score 1, we conclude that there are at most 4 arcs between them. But there's supposed to be $\binom{4}{2} = 6$ arcs!

In general, we get the following necessary conditions for an ascending sequence $p_1 \le p_2 \le \cdots \le p_n$ to be a score sequence of a tournament:

$$p_1 + p_2 + \cdots + p_k \ge \binom{k}{2} \qquad k = 1, \ldots, n-1$$
$$p_1 + p_2 + \cdots + p_n = \binom{n}{2}.$$

Theorem 1.3 (Landau)**.** *These conditions are also sufficient.*

We won't go into the details of the proof here. But the intuition for how to construct a tournament with this score sequence is to assume that the vertex with score p_n beats the vertices with scores $p_{n-1}, p_{n-2}, \ldots, p_{n-p_n}$, and repeat with the $n-1$ other vertices. This is very similar to how we proved the Havel–Hakimi theorem.

2 Hamiltonian paths and cycles

2.1 Hamiltonian paths

We didn't really talk about Hamiltonian paths earlier in this class, because we were pressed for time and these are objectively less interesting than Hamiltonian cycles. But a Hamiltonian path is just what you'd expect by analogy: it is a path that visits every vertex exactly once.

Why are we talking about them now? Because of this theorem:

Theorem 2.1. *Every tournament has a Hamiltonian path.*

Proof. Our tournament probably has cycles, so it probably doesn't have a topological ordering, but we can still do our best. Let $v_1, v_2, \ldots, v_n$ be an ordering of the tournament so that **as many as possible** of the arcs (v_i, v_j) satisfy $i < j$. Informally, the ordering has as many forward arcs as possible.[2]

In particular, for every $i = 1, 2, \ldots, n-1$, the arc between v_i and v_{i+1} has to be oriented (v_i, v_{i+1}) in this ordering. If we had the arc (v_{i+1}, v_i) instead, we could switch the order of v_i and v_{i+1} and increase the number of forward arcs: the arc (v_{i+1}, v_i) would become a forward arc, and nothing else would change. But we assumed we had as many forward arcs as possible, so this can't happen.

Therefore $(v_1, v_2, \ldots, v_n)$ is a Hamiltonian path. □

[2] This is called a **median order** of the tournament. It's very exciting for people who study tournament theory, but we will only get to see a glimpse of why.

Now we can go back and prove that the two notions of "transitive tournament" are equivalent. This is not a one-line proof in the notes, but that's only because I'm being nice to you and explaining the details; it is a one-line proof in spirit.

Corollary 2.2. *If a tournament is transitive (defined by the property that whenever (u, v) and (v, w) are arcs, so is (u, w)) then it is also acyclic.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be a Hamiltonian path in a transitive tournament.

We know that it has all the arcs (v_i, v_{i+1}) for $1 \le i \le n-2$. By transitivity applied to arcs (v_i, v_{i+1}) and (v_{i+1}, v_{i+2}), it also has all arcs (v_i, v_{i+2}) for $1 \le i \le n-2$. By transitivity again, we get arcs (v_i, v_{i+3}) for $1 \le i \le n-3$. Repeat this (technically we're giving a proof by induction) and we get arcs (v_i, v_j) for all $1 \le i < j \le n$.

Therefore $v_1, v_2, \ldots, v_n$ is a topological ordering, and the tournament is acyclic. □

2.2 Hamiltonian cycles

Hamiltonian cycles are a harder sell. Some tournaments don't have them; in particular, the transitive tournament doesn't have *any* cycles.

However, the exciting thing about tournaments is that here, we can write down a simple necessary and sufficient condition for a Hamiltonian cycle. (By "simple" I mean "can be checked much more easily than by a brute-force search for Hamiltonian cycles".)

Theorem 2.3 (Camion)**.** *A tournament on $n \ge 2$ vertices has a Hamiltonian cycle if and only if it is strongly connected.*

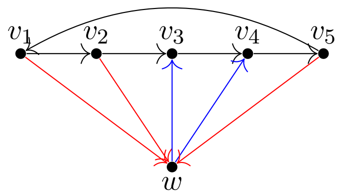
Proof. Why is strongly connected a necessary condition? Well, if you have a Hamiltonian cycle in a tournament, you can just follow it around to get from any vertex to any other vertex you like.

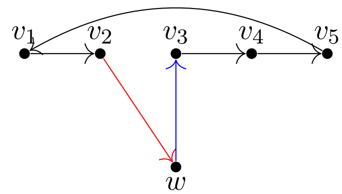
To prove that it's sufficient, we'll start small. We know that the transitive tournament is not strongly connected: it has a vertex of outdegree 0 that definitely can't reach any other vertex. So our strongly connected tournament is not transitive. Therefore it has three vertices u, v, w where transitivity fails: (u, v) and (v, w) are arcs, but we have the arc (w, u) rather than the arc (u, w).

This gives us a cycle (u, v, w, u) of length 3. Now we will simply work on making this cycle longer and longer until it's Hamiltonian.

Let's suppose our cycle at some step is $C = (v_1, v_2, \ldots, v_k, v_1)$. If we are not yet done, then there are some vertices of our tournament that are not in C.

Case 1. Suppose there is a vertex w outside C which has arcs both to and from vertices in C, as in the diagram on the left.

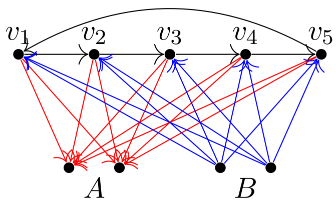


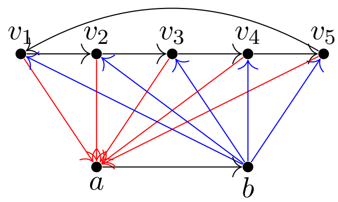


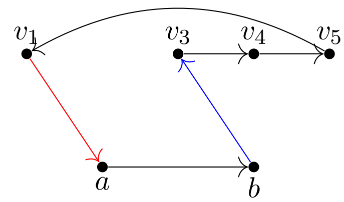
Then there must be a place along the cycle where the direction of the arcs switches: v_i has an arc **to** w, but v_{i+1} has an arc **from** w. (Maybe the switch happens from v_k to v_1 instead, but this is not substantially different.)

Then we can insert w into the cycle between v_i and v_{i+1}: $(v_1, v_2, \ldots, v_i, w, v_{i+1}, \ldots, v_k, v_1)$ is a bigger cycle. (See the second diagram above.)

Case 2. If case 1 doesn't happen, then the vertices outside C come in two types. There is a set A of vertices such that all the arcs between A and C point from C to A. Also, there is a set B of vertices such that all the arcs between A and C point from B to C. (See the first diagram below.)







Because the tournament is strongly connected, there must be a walk leaving cycle C to go to a vertex outside C, but then coming back to C. That walk must go from C to A (it can't go to B) but it can only come back to C from B, so eventually it must go from A to B somehow. In particular, there must be an edge (a, b) with $a \in A$ and $b \in B$. (See the second diagram above.)

It is now easy to extend cycle C pretty much any way we like. One way to get a cycle of length $k + 1$ (one longer than C) is $(v_1, a, b, v_3, \ldots, v_k, v_1)$. (See the third diagram above.)

In both cases, we can extend the cycle to make it longer. Eventually, we'll get a Hamiltonian cycle. $\square$

In the proof above, we went a bit out of our way to go from a cycle of length k to a cycle of length $k + 1$. (We could have gone from length k to length $k + 2$ in case 2; do you see how?) But our reward for that is the following:

Corollary 2.4 (Moon)**.** *Every strongly connected n-vertex tournament has a cycle of every length between 3 and n.*

Proof. In the proof, we start with a cycle of length 3 and end with a cycle of length n, so we encounter a cycle of every intermediate length along the way. $\square$