Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 23: Wrapping Up Matchings

November 4, 2021 *Kennesaw State University*

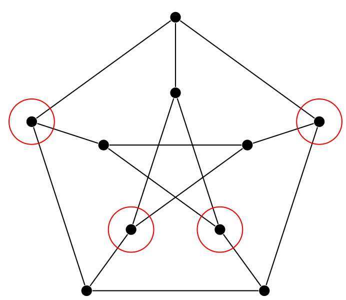
1 Four graph parameters

1.1 Definitions

In the recent lectures, we've looked at ways to find a maximum matching in a graph G. The number of edges in a maximum matching has a name: the **matching number** of G, appropriately enough. It also has its own special notation: we write $\alpha'(G)$ for the matching number of G.

Earlier this semester, we used notation like $\kappa(G)$ and $\kappa'(G)$ for the "vertex version" and "edge version" of an graph parameter—in that case, it was the connectivity. It should not be surprising that $\alpha'(G)$ is the "edge version" of a parameter: matchings in a graph are all about edges. What, then, is the "vertex version" $\alpha(G)$?

The number $\alpha(G)$ is the **independence number** of G: the size of a maximum independent set in G. An **independent set** $S \subseteq V(G)$ is a set of vertices in G such that there is no edge between any two vertices in S. For example, here is an independent set in the Petersen graph:



The precise way in which $\alpha'(G)$ is the "edge version" of $\alpha(G)$ is that a matching in G is exactly the same thing as an independent set in the line graph $L(G)$. Two edges share an endpoint in G (the thing a matching wants to avoid) precisely when the corresponding vertices are adjacent in $L(G)$ (the thing an independent set wants to avoid).

We will return to independent sets later in the semester, when talking about coloring graphs. For now, let's move on and talk about two more graph parameters.

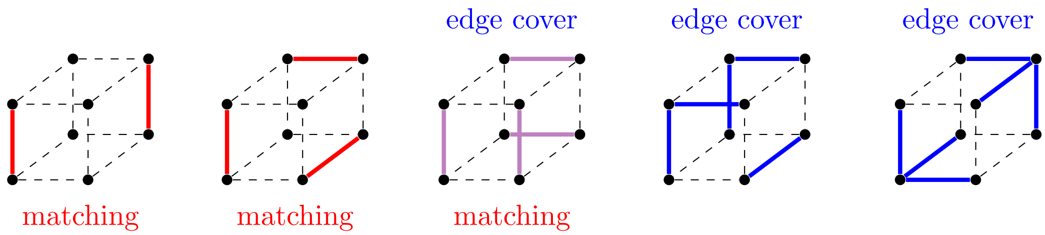
The **vertex cover number** $\beta(G)$ is the number of vertices in a minimum vertex cover of G. Now we can state König's theorem more concisely: it says that if G is a bipartite graph, then $\alpha'(G) = \beta(G)$. We've also seen that in general, $\alpha'(G) \leq \beta(G)$, but the two numbers might be different when G is not bipartite; for example, $\alpha'(K_{100}) = 50$ while $\beta(K_n) = 99$.

Finally, there is the **edge cover number** $\beta'(G)$: the "edge version" of $\beta(G)$. This is the number

[1] This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

of edges in a minimum edge cover of G, and an **edge cover** is a set of edges that cover all the vertices of G (that is, every vertex is an endpoint of one of the edges).

Edge covers and matchings are closely related: in a way, they are opposites. A matching of G can be viewed as a spanning subgraph in which every vertex has degree **at most** 1; an edge cover can be viewed as a spanning subgraph in which every vertex has degree **at least** 1. They meet in the middle: a perfect matching (if one exists) is also an edge cover. To illustrate this, here are some matchings and edge covers of the cube graph:



Unfortunately, $\beta'(G)$ is not the same thing as $\beta(L(G))$: for a set of edges in G to give us a vertex cover of $L(G)$, it has to be true that whenever two edges of G share an endpoint, one of them is included in the set. The line graph relationship only sometimes holds between the "vertex version" and "edge version" of an invariant; in other cases, this is a more informal analogy.

1.2 Relationships

What relationships can we find between these four invariants?

We might hope that the duality between matchings and vertex covers corresponds to a similar relationship between independent sets and edge covers. This is true:

Claim 1.1. *For any graph G, $\alpha(G) \le \beta'(G)$.*

Proof. Suppose that S is an independent set in G: a set of vertices with no edges between them. An edge cover of G must cover all the vertices of G, so in particular, it must cover all the vertices in S. But a single edge can only cover one vertex from S, because none of them are adjacent. Therefore every edge cover contains at least $|S|$ edges.

In particular, if S is the maximum independent set (with $|S| = \alpha(G)$) and L is a minimum edge cover (with $|L| = \beta'(G)$) then L must still contain at least $|S|$ edges: so $\alpha(G) = |S| \le |L| = \beta'(G)$. $\square$

Actually the two most closely related quantities among the four parameters are $\alpha(G)$ and $\beta(G)$. That's because they're secretly two ways of asking the same question!

If S is a set of vertices of G, let $\overline{S}$ denote its complement: the set of all vertices not in S.

Claim 1.2. *In any graph G, $S \subseteq V(G)$ is an independent set if and only if $\overline{S}$ is a vertex cover. In particular, if G has n vertices, then $\alpha(G) + \beta(G) = n$.*

Proof. To check that S is an independent set, we check that no edge has both of its endpoints in S. Equivalently, for every edge, at least one endpoint is in $\overline{S}$, instead. But that's exactly the condition that $\overline{S}$ covers every edge: that it's a vertex cover!

We maximize the size of S exactly when we minimize the size of $\overline{S}$. So a maximum independent set is the complement of a minimum vertex cover. Since $|S| + |\overline{S}| = n$, the number of vertices in G, we get $\alpha(G) + \beta(G) = n$ by taking the case where S is maximum and $\overline{S}$ is minimum. $\square$

It is far from true that the complement of any matching is an edge cover, and vice versa. For example, if a graph has any isolated vertices, then it has no edge cover: there is no edge that can cover an isolated vertex! For another example, consider the complete graph. A matching in K_n has at most $\frac{n}{2}$ edges, so the complement of a matching has $\frac{n(n-1)}{2} - \frac{n}{2}$ edges: almost all of them! But it doesn't take nearly that many edges of K_n to get an edge cover.

There is, nevertheless, a surprising relationship between $\alpha'(G)$ and $\beta'(G)$.

Theorem 1.3 (Gallai)**.** *If G is an n-vertex graph without isolated vertices, then $\alpha'(G)+\beta'(G) = n$.*

This is *not* something we naturally expect! Why is n, the number of vertices, the sum, even though both $\alpha'(G)$ and $\beta'(G)$ are counting edges? This is why this is a theorem and not a "Claim".

Proof. We will do two things.

1. First, we'll start with a matching M of size $\alpha'(G)$, and use it to construct an edge cover of size $n - |M|$. This shows that $\beta'(G) \leq n - \alpha'(G)$, since the edge cover we construct might not be the smallest.

2. Second, we'll start with an edge cover L of size $\beta'(G)$, and use it to construct a matching of size $n - |L|$. This shows that $\alpha'(G) \geq n - \beta'(G)$, since the matching we construct might not be the largest.

Together, these two inequalities will imply the equation we want.

For the first step, let M be a maximum matching. A matching is already a good start at an edge cover: with just $|M|$ edges, we've covered $2|M|$ different vertices! There are $n - 2|M|$ vertices left. For each of them, just pick an arbitrary edge that covers it. Together with these $n - 2|M|$ more edges, M becomes an edge cover, and the number of edges in it is $|M| + (n - 2|M|) = n - |M|$.

For the second step, let L be a minimum edge cover. The important thing to know about a minimum edge cover is that it's always a forest[2], because if there were a cycle, deleting any edge of that cycle would give you a smaller edge cover.

A forest with $|L|$ edges has $n - |L|$ connected components. In the case of an edge cover, none of the components can be isolated vertices, because then those vertices wouldn't be covered by any edges of L. So we can get a matching of size $n - |L|$ by picking an arbitrary edge from each connected component of the forest. $\square$

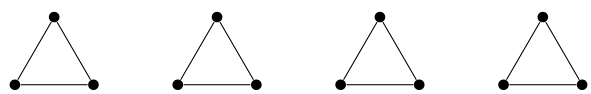
[2]More is true: it is always a "star forest". A star forest is one in which every component is a star, with one central vertex adjacent to many vertices of degree 1. But we don't need to prove this, because we don't need to use this fact in the theorem.

2 Odd fragments

We will end our discussion of matchings by look a bit at the obstacles that face matchings in arbitrary graphs.

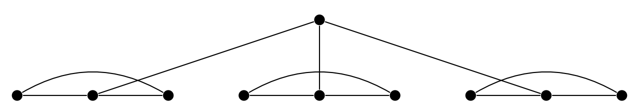
The new thing that we have to worry about is parity. The complete graph K_5 does not have a perfect matching, but it's not because it hasn't got enough edges! Rather, it's because it has an odd number of vertices. A matching of size 2 only covers 4 of the 5 vertices, but there's no matching of size 3, because that would need 6 vertices. So one vertex must be left over. (The same is true for any graph with an odd number of vertices.)

It's not just the total number of vertices that causes problems. Suppose we have a 12-vertex graph that consists of 4 copies of K_3 (for short, we call such a graph $4K_3$):



The matching number $\alpha'(4K_3)$ is only 4, because we can pick at most one edge from each component. It doesn't matter that the total number of vertices is even.

Finally, we can see parity becoming an issue even in connected graphs. Consider the graph below, with three copies of K_3 joined together by a 10^{th} vertex:



On its own, each copy of K_3 can only support one edge of a matching; a vertex will be left over. The 10^{th} vertex can rescue and pick up one of those leftover vertices—but only one. As a result, the matching number of this graph is 4.

This can generalize to more than one central vertex. You could imagine a graph with 500 odd fragments joined together by 100 central vertices. In any matching, every odd fragment would have at least one vertex left over (leaving 500 vertices uncovered). Then, the central vertices could (if we're lucky) be matched with 100 of those vertices. There would still be 400 uncovered vertices left over no matter what we do.

In fact, this is the central obstacle to matchings in general (non-bipartite) graphs. Let $o(G)$ denote the number of odd components in a graph G. Then we have:

Theorem 2.1 (Tutte)**.** *A graph G has a perfect matching if and only if, for every subset $U \subseteq V(G)$, we have $o(G - U) \le |U|$.*

In other words: for every set U, there are never more odd components in $G - U$ than can be "rescued" by matching them to every vertex of U.

It is important, when applying Tutte's theorem, to remember that $U = \varnothing$ is also a subset. The condition we get for $U = \varnothing$ is that G itself must have no odd components.

A generalization called the Tutte–Berge formula tells us the size of a maximum matching:

Theorem 2.2 (Tutte–Berge formula)**.** *If a graph G has n vertices, then*

$$\alpha'(G) = \frac{1}{2} \min_{U \subseteq V(G)} \{n - o(G - U) + |U|\}.$$

The logic here is that, for any set U, $n - o(G - U) + |U|$ is an upper bound on how many vertices can be covered by any matching. In every odd component of $G - U$, we will have at least one uncovered vertex—except for $|U|$ of them, which can be matched to a vertex of U. To convert from the number of covered vertices to the number of edges in the matching, we multiply by $\frac{1}{2}$.

What I've outlined is just the argument for why the condition in Tutte's theorem is necessary—and why the Tutte–Berge formula is an upper bound on $\alpha'(G)$. It takes much more work to show that the condition in Tutte's theorem is sufficient—and that $\alpha'(G)$ is really equal to the value given by the Tutte–Berge formula. We will not prove this theorem.