Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 24: Planar Graphs

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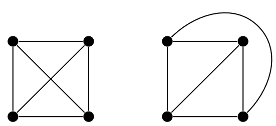
1 What are planar graphs?

Our next topic is understanding the question: when can we draw a picture of a graph in which none of the edges cross?

Sometimes answering this question is easy. For example, the graph on the left (the complete graph K_4) has two crossing edges, but we can fix this just by drawing one of the two edges differently:



Sometimes answering this question is hard. Both of the graphs below are drawn with many crossing edges. For one of them, this can be fixed; for there other, there is no way to avoid edges crossing.

However, it is far from obvious which graph has which property. We will need to develop some tools before we can answer this question.

1.1 Fine print

It's important to distinguish between "a graph that can be drawn in the plane with no crossings" and "a drawing of a graph in the plane with no crossings".

The first of these is a graph property. If G and H are isomorphic, and we can draw G in the plane with no crossings, then we can draw H in the plane with no crossings: just relabel the drawing of G. We say that a graph we can draw on the plane with no crossings is a **planar graph**.

A planar graph is still just an abstract object: a set of vertices and a set of edges. We haven't picked a particular drawing for it, and there could be many drawings that are different from each other in important ways. We say that a **plane embedding** of a graph G is a drawing of G in the plane with no crossings. (It is also very common to call this a "plane graph", but distinguishing between two different things called "planar graph" and "plane graph" could get confusing.)

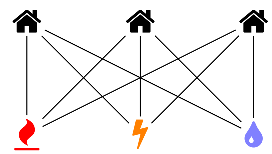
A planar graph, then, is a graph that has a plane embedding.

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

We should be careful when talking about properties of a plane embedding of G: they are not properties of the graph G itself, unless we can prove that **all** plane embeddings of G share those properties. We will see some examples and non-examples soon.

1.2 The utility graph

The three utilities problem is a classic graph theory puzzle. We have three houses that need to be connected to the water, gas, and electricity companies. Each of the 9 connections that need to be made is a separate pipeline (or wire, I guess); can this be done without any of the lines crossing?

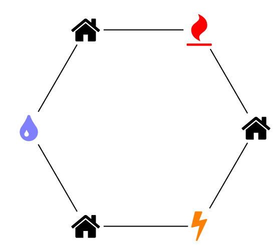


In other words, is the complete bipartite graph $K_{3,3}$ a planar graph?

(Due to the connection with this problem, $K_{3,3}$ is sometimes called the **utility graph**.)

Theorem 1.1. *The graph $K_{3,3}$ is not planar.*

Proof. The utility graph contains a Hamiltonian cycle with houses and utilities alternating. If a plane embedding of $K_{3,3}$ exists, that cycle must appear as a hexagon in that plane embedding. The hexagon could be very distorted; it could be concave and funny-shaped. However, we can always move parts of the embedding around to make it a regular hexagon, forming the following shape:



There are three more edges that need to be drawn. Each house still needs to be connected to the utility directly across it in the hexagon.

- If we draw two of the three edges inside the hexagon, they will have to cross. So at most one of the three edges can be inside the hexagon.
- If we draw two of the three edges outside the hexagon, they will have to cross. So at most one of the three edges can be outside the hexagon.

One edge is left that can neither be inside nor outside! Therefore $K_{3,3}$ is not planar. $\square$

There is a famous open problem in graph theory related to this one. Suppose we have $m > 3$ houses and $n > 3$ utilities: we are trying to draw the complete bipartite graph $K_{m,n}$. Of course, such a drawing contains a drawing of $K_{3,3}$, so it cannot be a plane embedding: some of the edges will cross. Can we minimize the number of crossings?

There is a drawing with

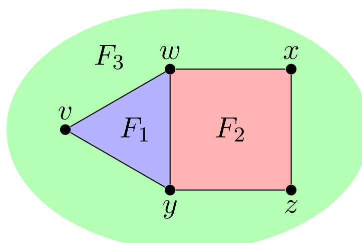
$$\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lfloor \frac{m-1}{2} \right\rfloor$$

crossings which is conjectured to be optimal. This has been verified for some small cases. The $m = n = 9$ case is the first for which we do not know if this solution is the best.

2 Faces

2.1 Faces and face lengths

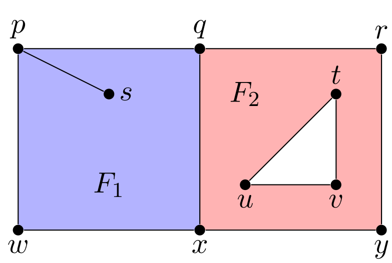
The edges of a plane embedding divide the plane into regions called the **faces**. There can be several "internal" or "bounded" faces, and there is always one "external" or "unbounded" face that extends outward infinitely far. Here is an example:



Here, F_1 and F_2 are the two internal faces, and F_3 is the external face.

In sufficiently nice embeddings, the boundary of a face is a cycle in the graph. Above, we see a plane embedding where the boundary of F_1 is the cycle (v, y, w, v), for example, and the boundary of F_3 is the cycle (v, y, z, x, w, v). In such cases, we say that the length $\text{len}(F)$ of a face F is the length of the cycle bounding it: therefore, $\text{len}(F_1) = 3$, $\text{len}(F_2) = 4$, and $\text{len}(F_3) = 5$ in the example above.

Sometimes, this can go wrong:



Although the cycle (p, q, x, w, p) is all that separates face F_1 from the outside world, we don't want to neglect edge ps as well. So we say that the boundary of F_1 is the closed walk (p, s, p, q, x, w, p), and $\text{len}(F_1) = 6$: the length of this closed walk.

For F_2, things are even worse: its boundary consists of the two cycles (q, r, y, x, q) and (t, u, v, t). We define $\text{len}(F_2) = 4 + 3 = 7$ in such a case: the total length of the two cycles. (This sort of thing only happens when the graph is not connected.)

Why do we define the length of a face in this way? We do it so that every edge contributes to the length of exactly two faces: the ones on either side of that edge. (Sometimes, as in the case of face F_1 and edge ps in the example above, this means contributing to the length of the same face twice.)

It follows that the faces and edges of a plane embedding obey the following formula: if there are m edges and k faces $F_1, F_2, \ldots, F_k$, then

$$\sum_{i=1}^{k} \operatorname{len}(F_i) = 2m.$$

This is one of two important identities regarding the faces of a plane embedding.

2.2 Euler's formula

The other important identity is Euler's formula, which tells us the number of faces in a planar graph.

Theorem 2.1 (Euler's formula)**.** *If a connected plane embedding has n vertices, m edges, and f faces, then*

$$n - m + f = 2.$$

Proof. We induct on the number of edges, m. Our base case is $m = n - 1$: the least number of edges in a connected graph.

In this case, the graph is a tree. All trees are planar (we don't need this for the proof, but it's a good exercise to do on your own). However, plane embeddings of trees only have one face: separating the plane into two regions requires a cycle. So in this case, $n - m + f = n - (n-1) + 1 = 2$.

Now suppose Euler's formula holds for all $(m-1)$-edge plane embeddings, and we have a plane embedding of a graph with m edges, where $m \geq n$. In this case, the graph has at least one cycle; let vw be an edge on a cycle.

In the plane embedding, the cycle separates the plane into two pieces (each of which might be made up of multiple faces). In particular, there are two different faces on the two sides of edge vw. So if we delete the edge vw, we are left with:

- A connected n-vertex plane embedding, since we deleted an edge on a cycle;
- Only $m - 1$ edges, since we deleted one;
- Only $f - 1$ faces, since the two faces on either side of vw merge into a single face.

By the inductive hypothesis, the new plane embedding satisfies Euler's formula: $n - (m-1) + (f-1) = 2$. This simplifies to $n - m + f = 2$, so Euler's formula also holds for the m-edge plane embedding we started with.

By induction, the formula holds for all plane embeddings. $\square$

Corollary 2.2. *If a plane embedding has n vertices, m edges, f faces, and k connected components, then*

$$n - m + f - k = 1.$$

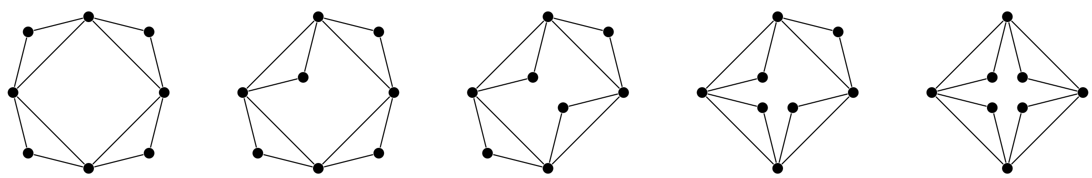
Proof. We can add $k-1$ edges to make the plane embedding connected. This does not change the number of faces: if we add an edge between two different connected components, it lies entirely in the face touching both compoents, and that face still remains a single face.

For the new plane embedding, Euler's formula says that $n - (m + k - 1) + f = 2$, which simplifies to $n - m + f - k = 1$. □

2.3 "Faces of a graph"

Faces are a property of a plane embedding. It is not okay to talk about the faces of a *planar graph*, because those faces are not necessarily determined before we have picked a plane embedding of a graph.

Consider the following five plane embeddings of the same graph G:



The faces here look very different! All of them have four faces of length 3, but:

- In the first plane embedding, there is a length-4 face and a length-8 face (the external face).
- In the second plane embedding, there is a length-5 face, and a length-7 face (the external face).
- In the third plane embedding, there are two length-6 faces.
- The fourth and fifth plane embedding matching the second and first in the face lengths. However, they swap which face is the external face! In the last plane embedding, there is a length-4 face and a length-8 face, but it's the length-4 face which is external.

Therefore it makes no sense to talk about the faces of a planar graph. If two people have the same graph, and both draw plane embeddings, their plane embeddings might not have the same faces, and they will be confused.

However, we can say two things:

1. All plane embeddings of the same graph have the same number of faces. This is because everything *except* for f in Euler's formula is determined by the graph, so f must be as well.
2. The sum of face lengths is determined by the graph, because it's equal to $2m$: twice the number of edges in the graph.