1 Lower bounds on chromatic number

Let’s summarize what we know about the chromatic number $\chi(G)$ of a graph $G$ so far:

- For any graph $G$, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree.
- If $G$ is a planar graph, then $\chi(G) \leq 4$. (We only proved that $\chi(G) \leq 6$; proving that $\chi(G) \leq 4$ is the much harder Four Color Theorem.)
- If $G$ is an interval graph, then $\chi(G) \leq \omega(G)$, where $\omega(G)$ is the clique number.
- If you’ve looked at the practice problems, you also know that if $G$ has $m$ edges, then $\chi(G) \leq 1 + \sqrt{2m}$.

These all have one thing in common: they are upper bounds on $\chi(G)$. That’s all we’re going to get from looking at variants of the greedy algorithm the way that we’ve been doing. Arguments like that are telling us that there is a way to color $G$ using some number of colors, and here’s how.

So how do we prove lower bounds on $\chi(G)$? This requires some kind of different argument: we need to show that colorings with a certain number of colors are impossible.

We will begin with two important lower bounds using quantities we’ve looked at before.

1.1 Lower bound via clique number

What’s the most straightforward graph that requires $k$ colors? It is the complete graph $K_k$: its $k$ vertices are all adjacent, so all of them need different colors.

Moreover, if a large graph $G$ contains a copy of $K_k$ inside it, then we know that $\chi(G) \geq k$ as well. Even the copy of $K_k$ inside $G$ needs $k$ colors: coloring the other vertices can only make things worse. (In general, if $H$ is a subgraph of $G$, then $\chi(G) \geq \chi(H)$, because to color $G$, we must first color $H$.)

We have a word for a copy of $K_k$ inside $G$: we called it a clique of size $k$. The size of the largest clique in $G$ is the clique number $\omega(G)$. We conclude:

**Theorem 1.1.** For any graph $G$, we have $\chi(G) \geq \omega(G)$.

This lower bound is the easiest of all lower bounds to see. In fact, it takes some work to convince yourself that $\chi(G)$ is not always the same as $\omega(G)$.

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1This document comes from the Math 3322 course webpage: [http://facultyweb.kennesaw.edu/mlavrov/courses/3322-spring-2022.php](http://facultyweb.kennesaw.edu/mlavrov/courses/3322-spring-2022.php)
Long odd cycles are the simplest example showing that $\chi(G)$ and $\omega(G)$ can be different. Any odd cycle like $C_5$ or $C_7$ or $C_{101}$ has chromatic number at least 3, because it is not bipartite. (In fact, because the maximum degree is 2, the chromatic number can’t be more than 3, so it is exactly 3.) However, with the exception of $C_3$, odd cycles only have clique number 2.

We will see more elaborate examples soon.

1.2 Lower bound via independence number

The independence number $\alpha(G)$ is the exact opposite of the clique number $\omega(G)$: it is the size of the largest set of vertices with no edges between them. So it may seem surprising that the independence number can also help us put lower bounds on the chromatic number.

The color classes of a coloring are the sets of vertices of each color. For example, if we color the vertices of a graph red, blue, and orange, the set of red vertices is a color class; the set of blue vertices is a color class; the set of orange vertices is a color class. Here is an example illustration:

\[ \begin{align*}
\text{R} & + \text{R} + \text{R} + \text{R} + \\
\text{R} & + \text{B} + \text{B} + \text{R} + \\
\text{B} & + \text{O} + \text{O} + \\
\text{O} & = \text{R} + \text{R} + \text{R} + \text{R} + \\
\text{R} & + \text{B} + \text{B} + \text{R} + \\
\text{B} & + \text{O} + \text{O} + \\
\text{O} &
\end{align*} \]

In a proper coloring, two vertices of the same color can’t be adjacent, and therefore the color classes are independent sets. In fact, that’s a characterization of proper colorings: they are partitions of the vertices of $G$ into independent sets. (Many important applications of graph coloring come from thinking of colorings in this way.)

This gives us another lower bound on chromatic number:

**Theorem 1.2.** If $G$ is an $n$-vertex graph with independence number $\alpha(G)$, then $\chi(G) \geq \frac{n}{\alpha(G)}$.

**Proof.** The independence number $\alpha(G)$ is the largest number of vertices in any independent set, so in particular every color class in a proper $k$-coloring has at most $\alpha(G)$ vertices. But the union of all $k$ colors must give us all $n$ vertices, so $k \cdot \alpha(G)$ must be at least $n$. Rearranging, we get $k \geq \frac{n}{\alpha(G)}$. \qed

1.3 How good are these bounds?

It is very easy to come up with examples of graphs that “fool” the lower bound $\chi(G) \geq \frac{n}{\alpha(G)}$: graphs that have very large chromatic number, even though $\alpha(G)$ is small. For example, consider the 7-sunlet graph, shown below:
The 7-sunlet graph consists of a 7-vertex clique on the inside, with a “ray” from each vertex of the clique to its own vertex of degree 1. Due to the 7-clique, this graph needs at least 7 vertices to color; in fact, once we color the central clique with 7 colors, it isn’t hard to color the rest of the graph without using any other colors. So the chromatic number is 7.

On the other hand, there are 14 vertices and a 7-vertex independent set: take all the outer vertices. So the bound of Theorem 1.2 only tells us that the chromatic number is at least 2: not very helpful!

We can make this example arbitrarily bad by generalizing from the 7-sunlet graph to the \( n \)-sunlet graph, built around a clique of size \( n \).

It is much harder to find examples where \( \chi(G) \) and \( \omega(G) \) are far apart. We will see two situations where this happens.

# 2 The Mycielski construction

The Mycielski construction is an iterative construction for building graphs where \( \chi(G) \) is high, but \( \omega(G) \) is low. In fact, the graphs we construct in this section will have \( \omega(G) = 2 \). A clique of size 3 is also called a **triangle**, because that’s what it looks like: 3 vertices with all 3 edges between them. So a graph with \( \omega(G) = 2 \) is often called **triangle-free**.

To find these graphs, we first define a new operation on graphs. Given a graph \( G \), the **Mycielskian** of \( G \) is a graph \( M(G) \) constructed as follows.

1. Start with a copy of \( G \), with vertices named \( v_1, v_2, \ldots, v_n \).
2. For each vertex \( v_i \), add a “shadow vertex” \( u_i \) adjacent to all of \( v_i \)’s neighbors in the copy of \( G \).
   (We never add edges between two different shadow vertices.)
3. Finally, add a vertex \( w \) adjacent to all the shadow vertices \( u_1, u_2, \ldots, u_n \).

Here is an example of this construction in action, with an arbitrary starting graph \( G \). (This is not the graph \( G \) we’ll ultimately want to apply the construction to.)

\[ \text{The original graph } G: \]

\[ \text{The shadow vertices:} \]

\[ \text{The final vertex } w: \]

**Claim 2.1.** If \( G \) is triangle-free, then so is \( M(G) \).

**Proof.** Where could we try to find a triangle in \( M(G) \), when there are no triangles in \( G \)?

If one vertex of the triangle were the final vertex \( w \), then the other two vertices would both have to be shadow vertices \( u_i, u_j \). This does not work, because \( u_i \) and \( u_j \) are not adjacent to each other.
Similarly, we cannot have a triangle with more than one shadow vertex in it. So the only hope for creating a triangle is to take one shadow vertex $u_i$, and two original vertices $v_j, v_k$. (You can see some triangles like this in the diagram above.)

But if all three of these vertices are adjacent, then $v_i$ is also adjacent to $v_j$ and $v_k$, so $v_i, v_j, v_k$ form a triangle as well! Therefore this cannot happen if $G$ is triangle-free: $M(G)$ must also be triangle-free.

**Claim 2.2.** $\chi(M(G)) = \chi(G) + 1$: the Mycielskian operation increases chromatic number by 1.

**Proof.** One direction of this claim (which we don’t particularly need in the end, but which we’ll do anyway because it’s easy) is to show that $\chi(M(G)) \leq \chi(G) + 1$: if we can color $G$ with $\chi(G)$ colors, we can color $M(G)$ with $\chi(G) + 1$ colors.

To do this, just take a proper $k$-coloring of $G$ and apply it to the copy of $G$ inside $M(G)$. Then, give every shadow vertex $u_i$ the same color as the color of $v_i$: this works fine, because it has all the same neighbors as $v_i$. Finally, give $w$ a new color we did not use in $G$. This also cannot create any conflicts, so we get a proper $(k + 1)$-coloring of $M(G)$.

The other direction is harder: given a proper $k$-coloring of $M(G)$, we must construct a proper $(k − 1)$-coloring of $G$. The algorithm to do so is this: given a proper $k$-coloring of $M(G)$, if any vertex $v_i$ has the same color as $w$, change it to have the same color as $u_i$ (which must be different from $w$’s color, because $u_i$ is adjacent to $w$). Here is an example:

You’ll notice that when we do this, the new coloring is no longer proper: the new color of $v_i$ might conflict with some of its shadow vertex neighbors. However:

**If $v_i$ is adjacent to $v_j$ and we recolor $v_i$, then $v_i$ and $v_j$ are different colors.**

That’s because $u_i$ was also adjacent to $v_j$ in the proper coloring we started with, so the color of $u_i$ (which is the new color of $v_i$) is different from the color of $v_j$. Also, none of the vertices we recolored were adjacent (because they were all the same color as $w$), so we don’t get any conflicts between two vertices that change colors.

Another way to say the bolded statement is that if we take the new coloring of $M(G)$ and only look at the copy of $G$ inside it, we get a proper coloring of $G$. That proper coloring uses one fewer color, because the color of $w$ no longer appears on any of $v_1, v_2, \ldots, v_n$. This gives us the $(k − 1)$-coloring of $G$ we wanted.

As a consequence, if we start with an arbitrary triangle-free graph, and apply the Mycielski construction over and over and over, we get a sequence of triangle-free graphs with growing chromatic number. This demonstrates that $\chi(G)$ might be much larger than $\omega(G)$.
Traditionally, we start with $K_2$: two vertices with an edge. Then $M(K_2) = C_5$: the 5-cycle. $M(C_5)$ is the Grötzsch graph, shown below:

The Grötzsch graph is the smallest triangle-free graph with chromatic number 4.

Collectively, the sequence of graphs $K_2, M(K_2), M(M(K_2)), M(M(M(K_2))), \ldots$ are sometimes called the Mycielski graphs.

3 Coloring random graphs

When we were looking at Ramsey numbers a few lectures ago, we saw that a randomly chosen graph is a good candidate for a graph $G$ where both $\alpha(G)$ and $\omega(G)$ are low.

Let’s return to this, but make it a bit more concrete. And to get a large-scale view of the problem, let’s consider graphs with $n = 1000000$ (a million) vertices.

There are many such graphs. To pick one of them at random, we flip a coin for each pair of vertices to decide if there is an edge between them. There is a mindbogglingly large number of graphs like this; every time you go to the trouble of flipping all $1000000^2$ coins to get a random 1000000-vertex graph, it is overwhelmingly likely that you are looking at a graph no human has ever seen before. But in some ways, these graphs are very predictable. They all have about $\binom{1000000}{2}$ edges, give or take a few million. And all of them have fairly small clique number and independence number.

Claim 3.1. If $G$ is a random graph chosen in this way, it is very unlikely that $\omega(G) \geq 40$.

Proof. There are $\binom{1000000}{40} \approx 1.22 \times 10^{192}$ ways to choose 40 of the vertices of $G$. Each one of those 40-vertex sets could be a 40-vertex clique, if we’re lucky.

It could be a 40-vertex clique, but that’s very unlikely. There are $\binom{40}{2} = 780$ edges between those 40 vertices, so we flip 780 coins to decide which edges between the vertices are present. In order for us to get a clique, all the coinflips have to go one way, which has a probability of $2^{-780} \approx 1.57 \times 10^{-235}$.

As an upper bound on the probability of getting a 40-vertex clique, it’s enough to just multiply these two numbers together. If there are about $1.22 \times 10^{192}$ potential 40-vertex cliques, and each one of them has about a $1.57 \times 10^{-235}$ chance of actually being a clique, then even if those are all disjoint events, the total probability of a 40-vertex clique is only $(1.22 \times 10^{192}) \cdot (1.57 \times 10^{-235})$ or about $1.93 \times 10^{-43}$.

That’s a tiny probability: we’re looking at an event that almost never happens. Our graph on 10000000 vertices almost never has any 40-vertex cliques. \qed

Similarly:
Claim 3.2. If $G$ is a random graph chosen in this way, it is very unlikely that $\alpha(G) \geq 40$.

Proof. The probability here is exactly the same as for cliques. For any given 40-vertex set, if we are flipping coins for each of the 780 edges, there is (again) a $2^{-780}$ probability that none of the edges are present.

So once again, there is at most a $1.93 \times 10^{-43}$ probability of getting a 40-vertex independent set. \qed

Let $G$ be a 1000000-vertex graph with $\alpha(G) \leq 40$ and $\omega(G) \leq 40$. What do we know about its chromatic number?

- Theorem 1.1, our bound via clique number, says that $\chi(G) \geq 40$.
- However, Theorem 1.2, our bound via independence number, says that $\chi(G) \geq \frac{1000000}{40} = 25000$. That’s much bigger!

In some sense, this result tells us that for almost all large graphs, Theorem 1.2 gives a much more useful bound than Theorem 1.1. Even though clique numbers often seem useful in small examples, those small examples are misleading: large graphs are very different.

This is not to say that Theorem 1.1 is useless. Not every graph we encounter is a randomly-generated graph. And we’ve already seen that for interval graphs, for example, the clique number always tells us the exact truth about the chromatic number.

In summary, both bounds are useful—and neither should be assumed to be correct without doing more investigation.
4 Practice problems

1. Let $G$ be the 25-vertex graph shown below:

(a) What is the clique number $\omega(G)$? What lower bound on $\chi(G)$ does it give?

(b) What is the independence number $\alpha(G)$? What lower bound on $\chi(G)$ does it give?

(c) What is the maximum degree $\Delta(G)$? What upper bound on $\chi(G)$ does it give?

(d) What is the actual chromatic number of $G$?

2. A wheel graph $W_k$ is the graph obtained from the cycle graph $C_n$ by adding a vertex adjacent to all vertices of the cycle.

(a) Draw a diagram of $W_5$.

(b) Show that $\omega(W_5) = 3$ but $\chi(W_5) = 4$.

(c) Generalize this construction to find a graph $G$ in which $\omega(G) = k$ but $\chi(G) = k + 1$, for every $k \geq 3$.

3. (a) Draw the graph $M(K_3)$.

(b) What are all the triangles in this graph?

Does $M(K_3)$ have any cliques with 4 or more vertices?

4. The Mycielski graphs have another interesting property. They are edge-critical: if you delete any edge, the chromatic number decreases.

(a) Show that $C_5$ is edge-critical.

(b) Show that if $G$ is edge-critical, then $M(G)$ is edge-critical. By induction, conclude that all Mycielski graphs are edge-critical.

(Another way to think about this definition: if $e$ is any edge of $G$ and $\chi(G) = k$, then there is a $(k - 1)$-coloring of $G$ which is almost a proper coloring, except that the endpoints of $e$ have the same color.)