Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 25: Planarity Testing

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1 Triangulations

1.1 The number of edges in a planar graph

Last time, we proved Euler's formula: in a connected plane embedding with n vertices, m edges, and f faces, $n - m + f = 2$. We also know that if we sum the lengths of all faces, then we get $2m$: twice the number of edges.

Except for some trivial cases with only 1 or 2 vertices, each face has at least 3 sides: there is at least a cycle separating it from another face, and that cycle by itself must have length at least 3. (We've also seen that weirder things can happen to the length of a face, but they only make the length longer.) Therefore if we sum the lengths of all faces, we get *at least* $3f$. This gives us an inequality between f and m (assuming $n \geq 3$):

$$2m \geq 3f.$$

We can use this inequality to prove the following theorem (which, intuitively, says: if there are too many edges, then we can't draw them all without crossings):

Theorem 1.1. *If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$.*

Proof. We may assume that G is connected; if not, we can add some edges to a plane embedding of G to connect it without ruining planarity, and that only increases m.

Then, combining Euler's formula $n - m + f = 2$ with the inequality $2m \geq 3f$, we get

$$f = 2 - n + m \wedge f \leq \frac{2}{3}m \implies 2 - n + m \leq \frac{2}{3}m$$

which we can rearrange to $\frac{1}{3}m \leq n - 2$, or $m \leq 3n - 6$. $\square$

This inequality lets us immediately conclude that some graphs are not planar. For example, the complete graph K_5 has $n = 5$ vertices and $m = 10$ edges. We have $10 > 3 \cdot 5 - 6$; therefore K_5 cannot have a plane embedding.

Note, however, that this is a **necessary** condition, not a **sufficient** one. If $m \leq 3n - 6$, we cannot conclude that a graph is planar! Here's a boring example: take K_5, and then add 95 isolated vertices. Here, $n = 100$ and $m = 10$, so m is much less than $3n - 6$; but the graph is still not planar. (We will see less boring examples later today.)

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

1.2 Looking at the extreme cases

Whenever we prove an inequality, a natural question to ask is: what can we say about the cases where equality holds? What kind of planar graphs have $m = 3n - 6$?

To draw conclusions about such graphs, we should look back at our proof, and look at every place where an inequality appeared:

1. We said "We may assume G is connected" on the basis that if it's, not, we can get a connected planar graph with the same number of vertices, but more edges.

   So if a planar graph satisfies $m = 3n - 6$, it *must* be connected.

2. Our inequality $m \leq 3n - 6$ came from the inequality $2m \geq 3f$. If we had $2m > 3f$, we'd have gotten $m < 3n - 6$, instead, by the same argument.

   So if a planar graph satisfies $m = 3n - 6$, it must satisfy $2m = 3f$.

3. Our argument for $2m = 3f$ came from an inequality we only stated in words: every face has length *at least* 3.

   So if a planar graph satisfies $m = 3n - 6$, then every face must have length *exactly* 3 (in every plane embedding).

Such a plane embedding (a connected one in which all faces are triangles) is called a **triangulation**. In fact, we can show that:

Corollary 1.2. *For a planar graph G with $n \geq 3$ vertices, the following are equivalent:*

(i) G has $3n - 6$ edges.

(ii) Every plane embedding of G is a triangulation.

(iii) G is a maximal planar graph: if we add any edge to G, it stops being planar.

Proof. What have we already proved here, and what's new?

We just now showed that (i) $\iff$ (ii): the only way to reach $3n - 6$ edges is if every plane embedding is a triangulation. On the other hand, if a plane embedding *is* a triangulation, then every inequality in the proof of our theorem is an equality, and we get $m = 3n - 6$.

We also have (i) $\implies$ (iii) just from the inequality $m \leq 3n - 6$. If a graph has exactly $3n - 6$ edges, and we add an edge, it has more than $3n - 6$ edges, so it can no longer be planar.

The new part is that (iii) also implies (i) and (ii): there are no maximal planar graphs that "get stuck" before becoming a triangulation with $3n - 6$ edges. We will show that (iii) implies (ii)...

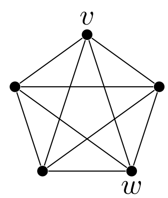
...by showing the contrapositive: if G has an embedding that's *not* a triangulation, then G is *not* a maximal planar graph. Essentially, the argument is that if a plane embedding of G has a face of length ≥ 4, then we can add an edge between two of its vertices, drawing it inside F. A few cases have to be considered to show that we can pick an edge that does not already exist *outside* F, but I'm skipping that in these notes, because it is tedious and not particularly educational. $\square$

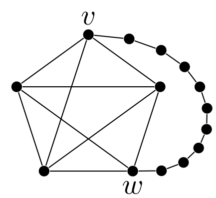
2 Subdivisions

2.1 Some more graphs that are not planar

Earlier today, we saw that K_5 was not planar, because it has too many edges.

We can get another example of a nonplanar graph as follows: take an edge vw of K_5, and replace it by a long $v - w$ path through entirely new vertices. The "before" and "after" of this procedure are shown below:





The second graph here has 9 more vertices and 9 more edges than K_5: $n = 14$ and $m = 19$. This comfortably satisfies the inequality $m \leq 3n - 6$.

However, the second graph is still not planar. Any plane embedding of the second graph would immediately give us a plane embedding of K_5: just replace the drawing of the long $v - w$ path by a drawing of the edge vw that traces out the same curve in the plane. Since K_5 is not planar, the second graph cannot be planar, either.

The general notion here is called a "subdivision". To **subdivide** an edge vw means to create a new vertex x, and replace edge vw by edges vx and xw. A **subdivision** of a graph G is a graph H obtained from G by subdividing edges some number[2] of times.

(In the example above, we subdivide edge vw, then subdivide the new edges created; every time we subdivide an edge along the $v - w$ path, it makes the path longer.)

For the same reasons as with the first example, if H is a subdivision of G, then they are either both planar or both not planar.

2.2 Kuratowski's theorem

So far, we have shown two graphs to be nonplanar: K_5 and $K_{3,3}$. As a consequence, a subdivision of K_5 or $K_{3,3}$ cannot be planar. Moreover, if a graph G *contains* a subdivision of K_5 or $K_{3,3}$ as a subgraph, then G cannot be planar: we can't even find a plane embedding of that subgraph of G, much less all of G.

The reason I emphasize K_5 and $K_{3,3}$ in particular is because of the following theorem:

Theorem 2.1 (Kuratowski)**.** *If a graph G is not planar, then G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

This is a sort of "guarantee of proof" theorem. In principle, it is easy to give a proof that G is a planar graph: just draw a plane embedding of G. (*Finding* the plane embedding may, admittedly,

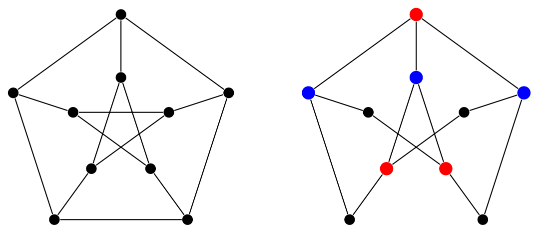
[2]To make the statement of Kuratowski's theorem simpler later, we say that G itself is a subdivision G: we happened to do the operation 0 times, which is also "some number".

be very hard. But at least we know that if G is planar, then such a demonstration exists.) However, proving that G is not a planar graph could be hard. Kuratowski's theorem says that if G is not planar, then we can always point out a subgraph of G that is a subdivision of $K_{3,3}$ or K_5, and then we have a proof that G is not planar.

Here is an example.

Claim 2.2. *The Petersen graph is not planar.*

Proof. Here is a subdivision of $K_{3,3}$ inside the Petersen graph:



By Kuratowski's theorem, the Petersen graph is not planar. □

Okay, but how do we find this subdivision? Some part of the process is creativity, but there are standard tricks we can try:

- Any subdivision of K_5 contains five vertices of degree 4: the vertices corresponding to the vertices of the original K_5. The Petersen graph does not have *any* such vertices, so it cannot contain a subdivision of K_5.

  This tells us that we must be looking for a subdivision of $K_{3,3}$, instead.

- In general, we'd want to pick the high-degree vertices to play the key roles in this subdivision. This doesn't help us here, though, because all vertices of the Petersen graph are identical.

- It can help to think of a subdivision of $K_{3,3}$ as picking vertices $v_1, v_2, v_3, w_1, w_2, w_3$, and then finding nine internally disjoint paths: a $v_i - w_j$ path for every i and j.

- To find *nine* such paths in a graph this small, most of the paths must be very short. So it makes sense, at least as a first try, to pick an arbitrary vertex to be v_1 and then its neighbors to be w_1, w_2, w_3. That takes care of the $v_1 - w_1$, $v_1 - w_2$, and $v_1 - w_3$ paths; we just have to locate v_2 and v_3.

3 Dual graphs

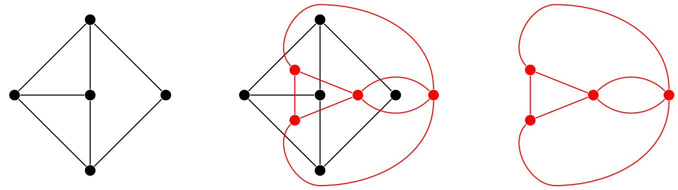
3.1 Definition

You may have noticed that the face length formula for plane embeddings is very similar to the degree sum formula for graphs:

$$\sum_{i=1}^{f} \text{len}(F_i) = 2m \qquad \sum_{i=1}^{n} \deg(v_i) = 2m.$$

This is not a coincidence! The face length formula is just the degree sum formula applied to a graph called the **dual graph** of the plane embedding, whose vertices are the faces $F_1, F_2, \dots$. We define the dual graph to have an edge F_iF_j whenever faces F_i and F_j touch along an edge in the plane embedding we started with.

Here is an example, with the original plane embedding in black and the dual graph in red. If we're careful, we can use the plane embedding of the original graph to find a plane embedding of the dual graph. Just place the dual vertex corresponding to face F_i somewhere in the interior of face F_i, and have the dual edges cross the edges of the original embedding.



However, you also notice from this picture that the dual graph is... not, strictly speaking a graph. It is a multigraph: when two faces touch along multiple edges, the dual graph has multiple edges between the corresponding vertices. Sometimes the dual graph has loops, when an edge has the same face on both sides.

It is also important to remember that saying "G^* is the dual graph of G" is not, strictly speaking, correct. The dual graph is defined in terms of a plane embedding of G; if we choose a different plane embedding, we may get a different dual graph.

3.2 Technical conditions

It turns out that several ways in which the dual graph G^* can be "nice" come from connectivity conditions on G. I will not prove these, but here is what we can say:

- If G is connected, then taking the dual of G^* will give us back a graph isomorphic to G.
- If G is 2-edge-connected (if $\kappa'(G) \geq 2$) then G^* will have no loops.
- If G is 3-edge-connected (if $\kappa'(G) \geq 3$) then G^* will be a simple graph.

Also, if $\kappa(G) \geq 3$ (if G is 3-vertex-connected), then the dual graphs we get from different embeddings of G will be isomorphic. This says that there is "essentially only one" plane embedding of a 3-connected planar graph G. Although we can fiddle with the embedding in many trivial ways (wiggle around some vertices or edges, or maybe even redraw it so that a different face is the external face), if two embeddings give isomorphic dual graphs, that means there's a correspondence between their faces, and the faces fit together in the same ways.