Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 26: Polyhedra and Factorizations

November 18, 2021 *Kennesaw State University*

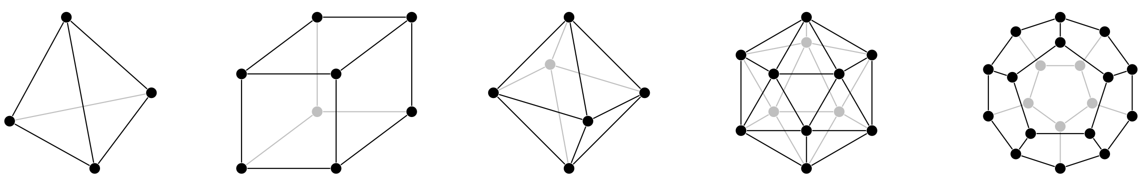
This is the last day of class before Thanksgiving break, so let's relax by doing graph theory with lots of pictures.

1 Polyhedra

1.1 Polyhedra and graphs

A **polyhedron** (plural: polyhedra) is the 3-dimensional version of a polygon: it's a 3D shape with polygonal sides. The sides meet at edges, and the edges meet at corners which are also called vertices. This is not a coincidence: if we have a polyhedron, we can form its **skeleton graph** whose vertices are the corners of the polyhedron, and whose edges are the geometrical edges of the polyhedron.

We have already seen the cube graph in many examples; this is the skeleton graph of a cube. As we will prove today, the cube is one of five Platonic solids: polyhedra whose faces are identical regular polygons, with the same number of faces meeting at each vertex. These are all shown below:



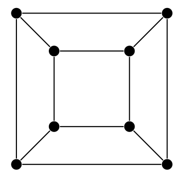
Tetrahedron Cube Octahedron Icosahedron Dodecahedron

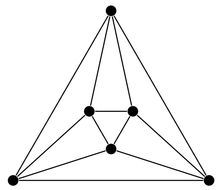
What's the connection to things we've done before in this class? Well, all five of these graphs are planar graphs. (More precisely: whenever we have a polyhedron with no holes in it—ones that can be drawn on the surface of a sphere—the skeleton graph is a planar graph.)

To get a plane embedding of one of these graphs, you should imagine taking one face of the polyhedron and stretching it out until everything else fits inside it. Alternatively, with care, you can draw the embedding directly, just by knowing how many sides the faces have, and how many faces meet at every vertex.

For example, here are the plane embeddings of the cube and octahedron, which are not too messy:

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php





1.2 Classifying the Platonic solids

Why are there only five Platonic solids? This is something we can prove from Euler's formula.

Theorem 1.1. *There are only five Platonic solids.*

Proof. We can describe a Platonic solid by a pair (p, q) where every face has p sides, and q faces meet at every vertex. Geometrically, we must have $p \geq 3$ and $q \geq 3$.

This lets us write down two equations for n (the number of vertices), m (the number of edges), and f (the number of faces).

- The graph is a q-regular graph, so by the degree sum formula, $nq = 2m$.
- Every faces has length p, so by the face length sum formula, $fp = 2m$.

We also have Euler's formula: $n - m + f = 2$. Replacing n by $\frac{2m}{q}$ and f by $\frac{2m}{p}$, we get

$$\frac{2m}{q} - m + \frac{2m}{p} = 2 \implies \frac{1}{q} - \frac{1}{2} + \frac{1}{p} = \frac{1}{m}.$$

From here, the constraint that lets us narrow down the pairs (p, q) is that $\frac{1}{m} > 0$. Therefore $\frac{1}{q} - \frac{1}{2} + \frac{1}{p} > 0$, or $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$.

How can we get a total bigger than $\frac{1}{2}$ here? Let's do casework on p:

- If $p = 3$ (every face is a triangle) then $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} = \frac{1}{6}$, so $q < 6$.

  We can have $q = 3$ (three triangles meet at every vertex), giving us the tetrahedron.

  We can have $q = 4$ (four triangles meet at every vertex), giving us the octahedron.

  We can have $q = 5$ (five triangles meet at every vertex), giving us the icosahedron.

- If $p = 4$ (every face is a square) then $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} = \frac{1}{4}$, so $q < 4$.

  We can have $q = 3$ (three squares meet at every vertex), giving us the cube.

- If $p = 5$ (every face is a pentagon) then $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} = 0.3$, so $q < \frac{1}{0.3} = 3\frac{1}{3}$.

  We can have $q = 3$ (three pentagons meet at every vertex), giving us the dodecahedron.

These are the only possibilities: if $p \geq 6$, then even $q = 3$ does not satisfy $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$. $\square$

In each of these cases, we can use Euler's formula to solve for n, m, and f. For example, in the case of the dodecahedron, we know that $3n = 2m = 5f$. We could write everything in terms of m:

$n = \frac{2}{3}m$, and $f = \frac{2}{5}m$. Putting this in Euler's formula, we get:

$$n - m + f = 2 \implies \frac{2}{3}m - m + \frac{2}{5}m = 2 \implies \frac{1}{15}m = 2 \implies m = 30.$$

So the dodecahedron has 30 edges. Since $n = \frac{2}{3}m$, the dodecahedron has $n = 20$ vertices; since $f = \frac{2}{5}m$, the dodecahedron has $f = 12$ sides.

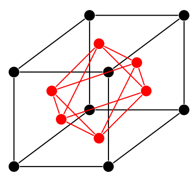
1.3 Dual polyhedra

Imagine taking any polyhedron P, and drawing a new polyhedron P^* by the following rule:

1. Put a point in the center of every face of P.
2. Draw a line segment connecting the centers of every two adjacent faces.

This is a geometric version of the dual graph of a plane embedding.

For example, if we draw a dual of the cube, we get the octahedron:



The dual of the octahedron gives us back the cube. The tetrahedron is its own dual, and the icosahedron and the dodecahedron are each other's duals.

2 Factorizations

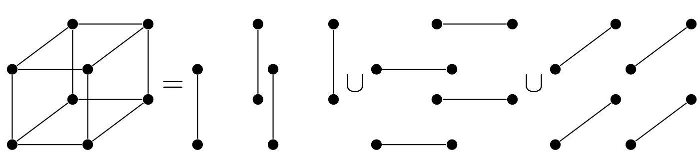
2.1 Defining 1-factors and 1-factorizations

When we say "degree" for the number of edges incident on a vertex, that goes back to older ways of thinking about graphs, which actually express graphs in terms of polynomials.

The same older ways give us the terminology "k-factor". A **k-factor** of G is a spanning subgraph of G (that is, a subgraph including every vertex of G) which is k-regular. We have already seen 1-factors under a different name: a 1-factor is just a perfect matching, because if F is a 1-factor of G, then every vertex of G is contained in exactly one edge of F.

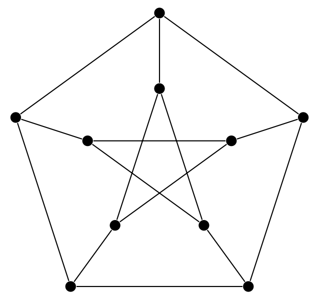
Where we have factors, we should also have factorizations. A **k-factorization** of G is a decomposition of G into k-factors. We have already seen the word "decomposition" before: it refers to splitting G up into pieces that share no edges. Here, each piece is a k-factor.

The most common kind of k-factorization is a 1-factorization: a decomposition into perfect matchings. For example, here is a 1-factorization of the cube:



(The connection to today's first topic is just coincidence; the cube graph is a convenient graph to use as an example.)

When does a graph have a 1-factorization? First of all, it must be regular (an r-regular graph might have a 1-factorization into r perfect matchings). It must also have a perfect matching to begin with. However, this is not enough: this is demonstrated by the Petersen graph.



The Petersen graph has a perfect matching: pair each vertex on the "outside" with the closest vertex on the "inside". However, this cannot continue: if this perfect matching is removed, what's left is two copies of C_5, and there is no second perfect matching to be found.

How can we be certain that this will never work, even if we pick a better starting matching? Well, if we delete any perfect matching from the Petersen graph, we're left with a 2-regular graph whose connected components are cycles. These cycles cannot be too short: the Petersen graph does not contain any 3-cycles or 4-cycles, so all the cycles must have length at least 5. They cannot be too long, either: the Petersen graph is not Hamiltonian. The only possibility this leaves is two 5-cycles: this is the result no matter which 1-factor we remove.

This is disappointing, but the good news is that for bipartite graphs, the result is better.

Theorem 2.1. *Every regular bipartite graph has a 1-factorization.*

Proof. Let G be a k-regular bipartite graph.

We proved earlier (in Lecture 22 from November 2nd) that such a graph G must have a perfect matching: a 1-factor, in the terminology of today's lecture. Pick any such matching and remove it from G. This reduces the degree of *every* vertex by 1, so we are left with a $(k-1)$-regular bipartite graph.

But now the same theorem applies to the result—so repeat until we run out of edges! (Formally, this should be an induction on k. The base case is $k = 1$, when G simply is a 1-factor. The argument above is the inductive step.) □

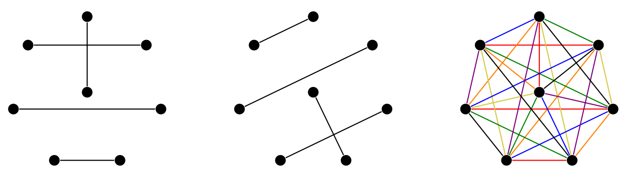
2.2 Factorizations of the complete graph

The most interesting 1-factorization to find is a 1-factorization of K_n. This has an immediate practical application: scheduling round-robin tournaments.

If you have n people that have all gathered to play chess (for instance), and you want everyone to play against everyone else, how do you do this as efficiently as possible? Well, you want to hold $\frac{n}{2}$ games at the same time, with $\frac{n}{2}$ pairs of people playing against each other. That's a perfect matching in K_n (which is not hard to find when n is even, just pair people off however you like). You want to schedule all $\binom{n}{2} = \frac{n(n-1)}{2}$ games in rounds like this one—and the partition of games into rounds is precisely a 1-factorization of K_n.

For even n, there is a nice "proof by picture". Draw a diagram of K_n in which $n-1$ vertices form a regular polygon, and the n^{th} vertex is at the center. To construct one of the matching, take an edge from the center vertex to one of the outside vertices together with all edges perpendicular to that one. As we go through all $n-1$ edges out of the center vertex, this gives us $n-1$ disjoint matchings that cover all the edges of K_n.

Below is an illustration for $n = 8$: two of the matchings, followed by a color-coded diagram of the entire 1-factorization.



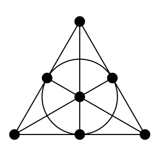
For odd n, there is no 1-factorization of K_n, because there are not even any 1-factors. However, we can get a factorization of K_n into n "nearly perfect" matchings, each of which covers all but one vertex. To find it, just delete a vertex from a 1-factorization of K_{n+1}. (This corresponds to a tournament scheduling strategy in which one person each round gets a "bye".)

There are other interesting decompositions of K_n to consider. For example, can we find a cycle decomposition of K_n in which every cycle is a 3-cycle?

Here, we have not one, but *two* divisibility constraints on n:

- In order to have any kind of cycle decomposition, we must have a graph where all degrees are even. Therefore n must be odd.
- In order to decompose a graph into 3-cycles, the total number of edges must be divisible by 3. For $\binom{n}{2} = \frac{n(n-1)}{2}$ to be divisible by 3, either n or $n-1$ must be divisible by 3.

This forces n to have the form $6k+1$ or $6k+3$ for some k: a number from the sequence $1, 3, 7, 9, 13, 15, \dots$. For $n = 1$ there is nothing to be done, and for $n = 3$ we just have a single 3-cycle. For $n = 7$ there is a related structure called the "Fano plane":



This is not a graph! There are 6 lines and a circle, each passing through 3 points (so they are not edges). But it has the property that any two points lie on exactly one common circle or line. So if we replace each line and circle by a 3-cycle, we get a 3-cycle decomposition of K_7.

In fact, it can be shown that:

Theorem 2.2. *For all n of the form $6k+1$ or $6k+3$ for some k, K_n has a 3-cycle decomposition.*