

Lecture 27: Graph Coloring

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Kennesaw State University

1 The chromatic number

Today we will talk about coloring graphs.

Informally, this means painting every vertex of a graph some color. We can formally define a **coloring** of G as a function $f : V(G) \rightarrow S$, where S is the set of colors: we call $f(v)$ the color of vertex v . Usually, it will not matter what S is: we can be poetic and pick a set like $S = \{\text{red, blue, purple, violet}\}$, or practical and pick a set like $S = \{1, 2, 3, 4\}$.

It *will* matter how large S is: how many colors we use. If $|S| = k$ (in which case, it is convenient to set $S = \{1, 2, 3, \dots, k\}$), we say that f is a k -coloring of G . We like to use as few colors as possible...

...but right now, we haven't said anything that requires us to use more than one color. We say that a **proper coloring** of G is a coloring in which adjacent vertices receive different colors. This goes back to the origins of graph coloring: coloring maps, in which adjacent regions should have different colors to distinguish them. However, graph coloring shows up in many applications: if edges represent "conflicts" between vertices (which is a very flexible interpretation), and we want to partition the vertices into non-conflicting groups, then we're looking for a proper coloring.

We say that G is **k -colorable** if it has a proper k -coloring. The **chromatic number** of G , denoted $\chi(G)$, is the minimum value of k for G is k -colorable. Once again, we're faced with an optimization problem, and so we can make some familiar observations:

- If we find any proper k -coloring of G , this is a proof by demonstration that G is k -colorable: that $\chi(G) \leq k$.
- Proving that $\chi(G) \geq k$ seems very hard. The brute-force approach is to show that none of the many many $(k - 1)$ -colorings of G are proper.

I am talking about graph coloring as though it is a new problem, but we have already seen one aspect of it near the beginning of the semester. A graph G is 2-colorable if and only if it is bipartite, and a bipartition is very nearly the same thing as a proper 2-coloring: just color vertices by which side of the bipartition they're on.

2 Greedy coloring

We are often interested in proving that all graphs of a certain type are k -colorable for some k . Here, it is not enough to draw a picture, because there could be infinitely many graphs of that type to

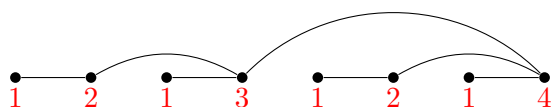
¹This document comes from the Math 3322 course webpage: <http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php>

color. Usually, the way we do this is to find an algorithm that tells us how to color the graphs we care about, and then prove that the algorithm never uses too many colors.

The simplest graph coloring algorithm is the greedy coloring algorithm. This does the following:

1. Number the vertices v_1, v_2, \dots, v_n in an arbitrary order. (We will usually illustrate this by drawing the graph so that the vertices are v_1, v_2, \dots, v_n from left to right.)
2. Color the vertices in that order, giving v_i the first color² that does not appear on any of its neighbors among v_1, v_2, \dots, v_{i-1} .

This is “greedy” because it does the first thing it can think of without thinking about future consequences. Sometimes this backfires. The graph below is a tree, so it is bipartite—and 2-colorable. But if we go from left to right, we get a 4-coloring (labels in red):



However, even this primitive strategy is enough to prove an upper bound on chromatic number.

Theorem 2.1. *For any graph G , $\chi(G) \leq \Delta(G) + 1$.*

Proof. We will prove that the greedy algorithm never uses more than $\Delta(G)+1$ colors. (Here, $\Delta(G)$ is the maximum degree of G .) In other words, if our colors come from the set $S = \{1, 2, \dots, \Delta(G)+1\}$, the greedy algorithm will never get stuck.

When we are coloring vertex v_i , it has at most $\Delta(G)$ neighbors among v_1, v_2, \dots, v_{i-1} . (It could have fewer for two reasons: it’s possible that v_i has fewer than $\Delta(G)$ neighbor, and it’s possible that not all of them come before v_i in the ordering we picked.)

Among those neighbors, there can be at most $\Delta(G)$ colors. Therefore there is at least one color from the set $S = \{1, 2, \dots, \Delta(G) + 1\}$ that does *not* show up on any of them. We can continue by giving v_i the first such color.

Since the greedy algorithm never fails to pick a color from S , at the end we will have a proper $(\Delta(G) + 1)$ -coloring of G . □

This bound is simple, but occasionally accurate. For example, it gives $\chi(K_n) \leq n$, which is accurate: the chromatic number of K_n really is n , because every vertex needs its own color.

Also, it correctly says that for any odd cycle C_{2k+1} , we have $\chi(C_{2k+1}) \leq 3$. Again, this is the best bound possible: odd cycles are not bipartite, so they’re not 2-colorable.

On the other hand, this bound also says that for any even cycle C_{2k} , we have $\chi(C_{2k}) \leq 3$. Well, actually, $\chi(C_{2k}) = 2$: we can 2-color an even cycle by alternating the two colors all the way around. So maybe this bound is not perfect.

²The “first color” assuming the colors are numbered $1, 2, 3, \dots$, so that they have a definite order.

3 Planar graphs

One of the big things we can do to improve the greedy algorithm is to be careful about the order in which we color the vertices. Unfortunately, finding the best order is just as hard as coloring the graph, which is very hard. But we can still sometimes prove better results with a better ordering.

One of the illustrations of this method is for planar graphs. The following is true:

Lemma 3.1. *Every planar graph G has minimum degree $\delta(G) \leq 5$.*

Proof. This is true for every planar graph with at most 6 vertices because at that point, you can't have any degrees bigger than 5.

For planar graphs with $n \geq 3$ vertices and m edges, we have the inequality $m \leq 3n - 6$. However, if every vertex had degree 6 or more, we would have $m \geq \frac{1}{2}(6n) = 3n$ by the handshake lemma, and we cannot have $3n \leq 3n - 6$. Therefore not all vertices have degree 6 or more: there must be a vertex with degree 5 or less. \square

Just having a small minimum degree would not help us. What *does* help us is that when you remove a vertex of minimum degree, we are left with a smaller planar graph, for which the following holds. This lets us pick a good vertex ordering:

Lemma 3.2. *Every planar graph G has a vertex ordering v_1, v_2, \dots, v_n in which each vertex is adjacent to at most 5 of the vertices that come before it.*

Proof. We will prove this by induction on n . When n is small (say, $n \leq 6$), any vertex ordering will do.

Assume that the lemma is true for all $(n-1)$ -vertex planar graphs, and let G be an n -vertex planar graph. By Lemma 3.1, G has a vertex v with $\deg(v) \leq 5$.

Apply the induction hypothesis to find a vertex ordering v_1, v_2, \dots, v_{n-1} of $G - v$. We can extend it to a vertex ordering of G by setting $v_n = v$. This vertex ordering does what we want for the first $n-1$ vertices by the induction hypothesis, and for the last vertex because $\deg(v) \leq 5$.

By induction, we can find such a vertex ordering for all n . \square

This proves the “six color theorem”:

Theorem 3.3. *Every planar graph G has $\chi(G) \leq 6$.*

Proof. Using the vertex ordering given by Lemma 3.2 and 6 colors, apply the greedy coloring algorithm to color G .

Since each vertex v_i is adjacent to at most 5 of the vertices that come before it, there will be at least one of the colors from $S = \{1, 2, 3, 4, 5, 6\}$ that does not appear on any of v_i 's already-colored neighbors. So we can always continue by assigning v_i some color from S . When we're done, we will have given G a proper 6-coloring. \square

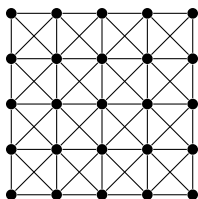
Much more is true! In fact, there is a famous theorem called the Four Color Theorem that $\chi(G) \leq 4$ for every planar graph G . However, this is much harder to prove, to the point that all known proofs require a computer to check many cases.

4 Cliques and independent sets

Now, we will look at a few lower bounds for chromatic number.

4.1 Cliques and clique number

Let G be the graph below. (If you want a name for it, it is $P_5 \boxtimes P_5$: the strong product of two paths. Don't worry about this notation.)



The maximum degree bound says that $\chi(G) \leq \Delta(G) + 1 = 9$, but that is too pessimistic. Actually, we can do better with the greedy coloring algorithm if we use a fairly natural ordering: color each row from left to right one at a time, starting at the top row and ending at the bottom row. With this ordering:

- We can see that we'll use at most 5 colors without even trying it: every vertex is adjacent to at most 4 vertices that come before it.
- But actually, if you do try it, you'll use only 4 colors.

Can we prove that there's no proper 3-coloring to conclude that $\chi(G) = 4$?

One way to prove this is to look at a "square" of 4 vertices in two adjacent rows and two adjacent columns. These will all be adjacent, so all of them need different colors, and that's already 4 colors used.

The general principle at work here is this: if G contains a copy of K_n inside it, then $\chi(G) \geq n$, because even the copy of K_n inside G needs at least n colors. (The other vertices of G not part of this copy of K_n might need more colors, or they might not—who knows?)

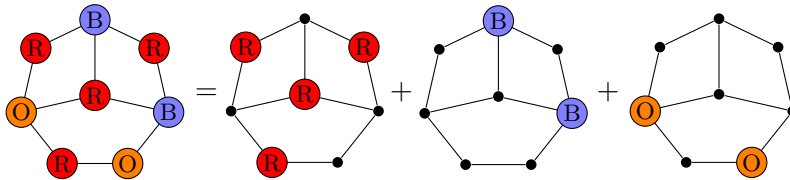
A copy of K_n for some n is called a **clique**: a set of vertices that are all adjacent to each other. (This is the opposite of an independent set, which is a set of vertices without any edges between them.) We define the **clique number** $\omega(G)$ to be the number of vertices in the largest clique in G .

Phrased formally, we have:

Theorem 4.1. *For any graph G , we have $\chi(G) \geq \omega(G)$.*

4.2 Color classes and independence number

The **color classes** of a coloring are the sets of vertices of each color. For example, if we color the vertices of a graph red, blue, and orange, the set of red vertices is a color class; the set of blue vertices is a color class; the set of orange vertices is a color class. Here is an example illustration:



In a proper coloring, two vertices of the same color can't be adjacent, and therefore the color classes are independent sets. In fact, that's a characterization of proper colorings: they are partitions of the vertices of G into independent sets. (Many important applications of graph coloring come from thinking of colorings in this way.)

This gives us another lower bound on chromatic number:

Theorem 4.2. *If G is an n -vertex graph with independence number $\alpha(G)$, then $\chi(G) \geq \frac{n}{\alpha(G)}$.*

Proof. The independence number $\alpha(G)$ is the largest number of vertices in any independent set, so in particular every color class in a proper k -coloring has at most $\alpha(G)$ vertices. But the union of all k colors must give us all n vertices, so $k \cdot \alpha(G)$ must be at least n . Rearranging, we get $k \geq \frac{n}{\alpha(G)}$. \square

4.3 Which is better?

Neither bound is perfect. If you look at the graph in subsection 4.1, and try to apply Theorem 4.2 to it, you'll only get $\chi(G) \geq \frac{25}{9} \approx 2.78$, even though the clique number shows that $\chi(G) \geq 4$. On the other hand, if you look at the graph in subsection 4.2, and try to apply Theorem 4.1 to it, you'll get a lower bound of 2 (because there are no copies of K_3 in the graph), even though the graph is not 2-colorable.

In the next (and final) lecture, we will look at when these bounds are good—and when they are bad.