Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 3: Proof techniques

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1 Unpacking definitions

There's two steps to proving a theorem. The first step is what I call "unpacking definitions". In this step, we fill in the initial steps of the proof by looking at the definitions of the things we're talking about, and the structure of the statements we're proving.

This is not straightforward—it's a skill to be learned before you can write proofs—but once you learn to do this, you will be able to do this for any theorem, no matter how hard it is to prove. Once that is done, though, you still have to do the second step: filling in the gaps.

Let's look at "unpacking definitions" using an example from the previous lecture:

Theorem 1.1. *Let G be a graph, and let $v \sim w$ if there is a $v - w$ walk in G. Then the relation $\sim$ is an equivalence relation on $V(G)$.*

The definition of an equivalence relation has three parts, so right away, we know that our proof is going to have three parts: we are going to prove that $\sim$ is reflexive, symmetric, and transitive.

Let's focus on one of those: proving that $\sim$ is symmetric. This is more complicated. The definition of a symmetric relation is "for all v, w, if $v \sim w$, then $w \sim v$". How does this help us structure our proof?

- The definition begins by saying "for all v, w". This means that we are given v and w (in this case, vertices of G) with *no* control over what they could be. We have to prove that the definition holds without any assumptions about v and w.
- The definition is an if-then statement. The default way to prove this is to assume the hypothesis (that $v \sim w$) and use this to prove the conclusion (that $w \sim v$).

This is still not entirely unpacked. The statements "$v \sim w$" and "$w \sim v$" have a definition given in the theorem itself. So what we're assuming is that there is a $v - w$ walk, and what we're proving is that there is a $w - v$ walk.

If we're assuming that something exists, it helps to give it a name. So let's refine our assumption. We're assuming that there is a sequence $(v_0, v_1, v_2, \dots, v_\ell)$ that is a $v - w$ walk in G. This (by definition of a walk) tells us three things: that $v_0 = v$, that $v_\ell = w$, and that $v_i v_{i+1}$ is an edge of G for $i = 0, \dots, \ell - 1$.

Our goal is to prove that there is a $w - v$ walk. To prove that something exists, we have to construct it, and that's a step that unpacking definitions can't help us with. We have to come up with an idea for what the $w - v$ walk should be.

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2024.php

But once we do have that idea, we can say what our next steps should be. Using our idea, we define a sequence of vertices that we intend to show is a $w - v$ walk. Then we check that it satisfies the definitions: it starts at w, it ends at v, and consecutive vertices in the sequence are adjacent. When that's done, we will have completed our proof.

To summarize, here is the "scaffolding" of a proof that $\sim$ is symmetric.

Proof. Suppose that $v \sim w$: that there is a $v - w$ walk in G. Our goal is to prove that there is also a $w - v$ walk in G.

Let $(v_0, v_1, v_2, \dots, v_\ell)$ be a $v - w$ walk (with $v_0 = v$ and $v_\ell = w$). Then define a new sequence of vertices by ...

This sequence of vertices starts at w, ends at v, and consecutive vertices in the sequence are adjacent because ...

Therefore it is a $w - v$ walk. Therefore a $w - v$ walk exists, so $w \sim v$. Since we assumed $v \sim w$ for arbitrary vertices v and w, and proved $w \sim v$, we conclude that $\sim$ is symmetric. □

The gaps with "..." are where we need to come up with some kind of idea: how exactly do we construct the $w - v$ walk? This can come from intuition about the problem, or by looking at some examples and noticing a pattern.

2 Quantifiers: "there exists" and "for all"

Lots of theorems and definitions in graph theory (and other areas of math) have the form "There exists an X such that property Y holds" or "For all X, property Y holds". Sometimes, these things appear as part of a more complicated statement. Sometimes, a "for all" is implied: for example, we might say "If $v \sim w$, then $w \sim v$" and mean "For all v and w, if $v \sim w$, then $w \sim v$".

We've already seen a bit of how these work in an example, but let's summarize:

- If you're assuming that something exists, you get to define it right away and start doing things with it. (We saw this with our assumption "there exists a $v - w$ walk" earlier.)

  This is often very powerful: it gives you more to work with.

- If you're trying to prove that something exists, you should construct an example (as we did with the $w - v$ walk). The word "construct" might sometimes mean you can write down exactly what it is. But it often means we need to explain a rule for how to build the thing we want, using the assumptions we have.

- If you're assuming that a property holds for all X, then we get to invoke this assumption whenever we see an X. But this is often hard to start writing a proof with, because we have to encounter an X somewhere, before we can do anything with it. Be on the lookout for objects you can apply this assumption to!

- To prove that a property holds for all X, you start with an arbitrary X, and try to prove that it has the property you want. We have no control over the X we are given, so we can't make any additional assumptions about it.

The negation of a "for all" statement is a "there exists" statement. For example, the negation of "all triangle-free graphs are 3-colorable" is "there exists a triangle-free graph that is not 3-colorable".[2]

The negation of a "there exists" statement is a "for all" statement. For example, the negation of "there is a connected graph with 10 vertices and 8 edges" is "all graphs with 10 vertices and 8 edges are not connected".

This property of negations sometimes lets us pick and choose which kind of statement we want to work with. That's because proving "If X, then Y" is the same as proving "If not Y, then not X". Or, we can use a proof by contradiction.

One final note—if something doesn't exist, then we consider "for all" statements about it to be true. For example, the statement "all $v - w$ paths have even length" is true in a graph which has no $v - w$ paths at all! That's because the negation of this statement is "there is a $v - w$ path which does not have even length" which is clearly false in a graph which has no $v - w$ paths.

3 Optimization problems

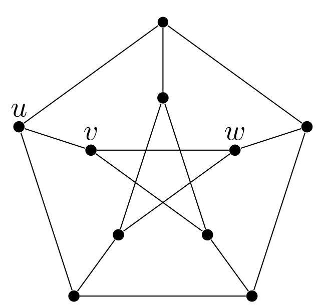
A lot of definitions in graph theory involve solving an optimization problem: they are phrased in terms of the biggest or smallest thing of a certain type. We've already seen two examples:

- The distance $d(u, v)$ between u and v is the length of the shortest $u - v$ path.
- The diameter of a graph G is the longest distance between any two of its vertices (that's two optimization problems in one)!

So a statement like "the distance between u and v is 5" has two parts to it:

- There is, actually, some $u - v$ path of length 5. (By itself, this proves that $d(u, v) \leq 5$.)
- There is no $u - v$ path of length 4 or less: all $u - v$ paths have length at least 5. (By itself, this proves that $d(u, v) \geq 5$.)

Let's begin with a concrete example. Consider the graph below (it's called the Petersen graph, and we will see it again):



Suppose that we want to prove that $d(u, w) = 2$. Two steps are necessary:

[2] Don't worry about what this means just yet, but we *will* see which of these two statements is true later in the semester.

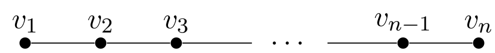
1. First, we prove that $d(u, w) \leq 2$. This involves constructing a path of length 2 that starts at u and ends at w. Such a path is (u, v, w).
2. Second, we prove that $d(u, w) \geq 2$. With length 2, this is not too bad: all we have to do is rule out $d(u, w) = 0$ and $d(u, w) = 1$.

   We can't have $d(u, w) = 0$ because a path of length 0 starts and ends at the same vertex, but $u \neq w$.

   We can't have $d(u, w) = 1$ because a path of length 1 starting at u and ending at w could only be the sequence (u, w). But this is not a path, because u and w are not adjacent.

Let's look at a more complicated and abstract example:

Theorem 3.1. *The path graph P_n shown below has diameter $n - 1$.*



Proof. First, we show that the diameter of this graph is at least $n - 1$. This requires showing that there exists a pair of vertices u, v with $d(u, v) = n - 1$. Actually, showing that there exists a pair of vertices u, v with $d(u, v) \geq n - 1$ is enough, saving us some work.

This pair of vertices will be v_1 and v_n. What we want to prove about them is that there is no $v_1 - v_n$ path of length $n - 2$ or less. Informally: in k steps from v_1, we can only get as far as v_{k+1}, so if $k < n - 1$, we can't reach v_n. Formally, we should prove this by induction on k, which we'll review next week.

Second, we show that the diameter of this graph is at most $n - 1$. This requires showing that for all pairs of vertices u, v, $d(u, v) \leq n - 1$: there exists a $u - v$ path of length $n - 1$ or less.

So let's construct a $u - v$ path that's not too long. There are two cases:

- If $u = v_i$ and $v = v_j$ with $i < j$, then our path will be $(v_i, v_{i+1}, \ldots, v_{j-1}, v_j)$. This has length $j - i < n - 1$.
- If $u = v = v_i$, then our path will be (v_i). This has length $0 < n - 1$.
- If $u = v_i$ and $v = v_j$ with $i > j$, then our path will be $(v_i, v_{i-1}, \ldots, v_{j+1}, v_j)$. This has length $i - j < n - 1$.

In both cases, we've shown that $d(u, v) \leq n - 1$, so the diameter of P_n is at most $n - 1$. $\square$

4 The extremal principle

Let's begin by proving a theorem I mentioned in the previous lecture.

Theorem 4.1. *Let v, w be two vertices of a graph G. If there is a $v - w$ walk in G, then there is a $v - w$ path in G, as well. Moreover, the shortest $v - w$ walk is always a path.*

Proof. Let $(v_0, v_1, \ldots, v_\ell)$ with $v_0 = v$ and $v_\ell = w$ be the shortest $v - w$ walk. We will prove that it's a path, which proves both parts of the theorem.

Suppose that it's not a path: that some vertices repeat. In particular, let's suppose that $v_i = v_j$ with $i < j$. Then the following is also a $v - w$ walk:

$$(v_0, v_1, \ldots, v_i, v_{j+1}, \ldots, v_{\ell-1}, v_\ell).$$

(We should check that it satisfies the definition. In fact, any two consecutive vertices in this walk were also consecutive in the walk we started with, so they are adjacent. This is not as obvious for the pair (v_i, v_{j+1}), but $v_i = v_j$, so this is the same as the pair (v_j, v_{j+1}).)

However, the length of this new walk is $\ell - (j - i)$, which is less than ℓ. So we've found a $v - w$ walk shorter than the $v - w$ walk we started with! That's a contradiction, because we assumed that we took the shortest $v - w$ walk there is. So our assumption that the walk was not a path is false: the walk *is* a path. □

This technique is called the "extremal principle". To prove that a $v - w$ walk that's also a path exists, we picked an *extremal* $v - w$ walk: one that's best by some metric. In this case, it was the shortest. Then, we reason: "The extremal thing we chose must have the property we want. Suppose it didn't. Then we could improve it to make it even better, which contradicts our initial assumption that it was the best."

In this case, the use of the extremal principle was baked into the theorem's statement. But taking the shortest $v - w$ walk is also the easiest way to prove the first part of the theorem: ''If there is a $v - w$ walk in G, then there is a $v - w$ path in G, as well." So sometimes, we can apply this technique even when there's nothing in the theorem telling us to use it.

Be careful! Before we can pick the "best" out of a set of objects, we have to know two things. First, we must know that the set isn't empty. (In this case, we had to know that there was a $v - w$ walk before we could pick the shortest $v - w$ walk.) Second, we must know that there *is* a "best". For integer quantities like distance, it's enough to know that there is a limit to how good things can get. (In this case, the limit is that all walks have a nonnegative length.) If there were no limit, it would be possible that there's no best object; no matter which one we pick, there was a better one we missed.

The theorem is also worth mentioning for other reasons. It tells us that two things are equivalent, even though one has fewer assumptions built in. That's how we use this theorem:

- If we are trying to prove that there is a $v - w$ path in G, it's enough to prove that there is a $v - w$ walk (which requires less work). Then, we can invoke this theorem.
- If we can assume that there is a $v - w$ walk in G, then we can strengthen that assumption to have a $v - w$ path, by invoking this theorem. (This assumption is more powerful: we get to say that the vertices of the path are all different, "for free".)

This theorem lets us always prove the thing that's easier to prove, and assume the thing that is more informative.

5 Practice problems

1. Here are some statements from topics we will later see in graph theory.

   For each one, rewrite the sentence to make the quantifiers and logical implications inside it explicit. Do not worry about looking up the terminology you haven't learned yet.

   Example: "Every connected graph has a spanning tree."

   Example solutions:

   - *"For all graphs G, if G is connected, then G has a spanning tree"*
   - *"For all graphs G, if G is connected, then there exists a graph H such that H is a spanning tree of G."*

   (a) Any two isomorphic graphs have the same number of edges.

   (b) Any two embeddings of a given planar graph have the same number of faces.

   (c) A graph with n vertices is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater.

   (d) Among triangle-free graphs (graphs with no 3-vertex, 3-edge subgraphs), there are graphs with arbitrarily large chromatic number.

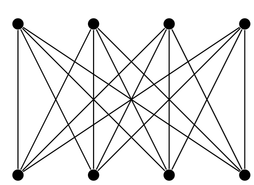
2. Find the error in this proof of the statement "Graphs never have edges".

   > Let G be any graph, and let v be an arbitrary vertex. Let $(v, v_1, v_2, \ldots, v_\ell)$ be the longest walk beginning at v.
   >
   > If there is an edge vw with endpoint at v, then the walk $(v, w, v, v_1, v_2, \ldots, v_\ell)$ is a longer walk beginning at v. That's a contradiction, because we assumed that we took the longest such walk. So the assumption that the edge vw exists is incorrect: there is no edge with an endpoint at v.
   >
   > Since v was arbitrary, G has no edges with an endpoint anywhere: no edges at all!

3. The Ramsey number $R(s,t)$ is defined as "The least positive integer n such that every n-vertex graph contains either a clique of size s or an independent set of size t". We will learn more about these concepts later on in the semester; for the purposes of this problem, you don't need to have any idea what a "clique" or "independent set" is.

   For now, write a scaffolding for a proof of the statement "$R(3,3) = 6$". That is, write a proof, but skip the parts where you actually need to have a clever idea about cliques or independent sets. Try to include as much detail as possible, otherwise!

4. Prove that the graph shown below has diameter 2:



5. A "P_3-free graph" is a graph that does not have as an induced subgraph. In other words, if you pick 3 vertices inside a P_3-free graph, there will never be exactly 2 edges between them: there could be 0 edges, a single edge, or all 3 edges.

   (a) Let G be a P_3-free graph, and let $v \sim w$ if v and w are adjacent (if $vw \in E(G)$).

   Prove that $\sim$ is an equivalence relation on $V(G)$.

   (b) What does this tell us about the structure of G? Describe what a connected component of G can look like.