Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 6: The degree of a vertex

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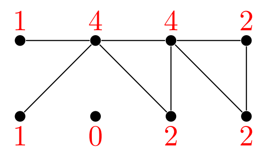
1 Degrees and the handshake lemma

First, some definitions. The **degree** of a vertex v in a graph G is the number of edges that have v as an endpoint. (We also say such edges are **incident on** v.) We write the degree of v as $\deg(v)$. If we have multiple graphs that share the vertex v for any reason, we might write $\deg_G(v)$ to specify that we mean the degree in G.

Here is a graph with vertices labeled according to their degree:



There's a lot of associated terminology. A vertex with degree 0 is called an **isolated vertex**, and a vertex with degree 1 is sometimes called a **leaf**. A graph G has a **maximum degree** (the largest degree of any vertex) and a **minimum degree** (the smallest degree of any vertex).

We write $\Delta(G)$ for the maximum degree and $\delta(G)$ for the minimum degree. Graph theory uses a lot of Greek letters for properties of graphs; this is only the beginning.

The first tool we'll need to make use of degrees is the Handshake Lemma (also known as the degree sum formula).

Lemma 1.1. *In any graph G, the vertex degrees add up to twice the number of edges:*

$$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)|.$$

Proof. Many proofs exist; for the sake of practice, let's do a proof by induction. We will prove that for any graph with m edges, the sum of degrees is $2m$, by induction on m.

The base case is $m = 0$. Here, we have a graph with no edges. No matter how many vertices we have, their degrees are all 0, the sum of the degrees is 0, and $2m$ is also 0.

Assume that the degree sum formula holds for all $(m-1)$-edge graphs. Let G be a graph with $m \geq 1$ edges, and let xy be any edge of G. We can apply the inductive hypothesis to $G - xy$ (the graph we get by deleting edge xy from G), a graph with $m - 1$ edges.

[1] This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2023.php

What is the relationship between the degrees of G and the degrees of $G - xy$? Both x and y have one extra incident edge in G they don't have in $G - xy$: edge xy itself. So

$$\deg_G(x) = \deg_{G-xy}(x) + 1 \text{ and } \deg_G(y) = \deg_{G-xy}(y) + 1.$$

For any other vertex v, $G - xy$ and G have the same number of edges, so we have

$$\deg_G(v) = \deg_{G-xy}(v).$$

Also, $G - xy$ and G have the same set of vertices. So if we add up the vertex degrees in $G - xy$ and G, the result is that

$$\sum_{v \in V(G)} \deg_G(v) = \left(\sum_{v \in V(G-xy)} \deg_{G-xy}(v) \right) + 2.$$

Applying the inductive hypothesis, we get that the degree sum in $G - xy$ is $2(m-1)$, so the degree sum in G is $2(m-1) + 2 = 2m$.

By induction, the degree sum formula holds for all graphs. $\square$

Let's answer some quick questions using the handshake lemma!

Q1. Is it possible to have a 2021-vertex graph in which every vertex has degree 3?

A1. No: then the degree sum would be $2021 \cdot 3 = 6063$, so we would have $\frac{6063}{2} = 3031.5$ edges, which is impossible.

Q2. A soccer ball has 12 black pentagonal panels (and some white hexagonal panels I'm too lazy to count). Panels are stitched along their edges, and meet at corners; at each corner, a pentagon and two hexagons meet. How many edges are there where two panels meet?

A2. Each black pentagon has 5 corners, which will be the $12 \cdot 5 = 60$ vertices of our graph; the edges will be the edges where panels meet. Here, each vertex has degree 3, so the sum of degrees is $60 \cdot 3 = 180$, and there are 90 edges.

(Some more logic can convince you that the number of hexagons is $\frac{1}{3}$ the number of vertices, or 20; we'll return to this later in the semester.)

Q3. Suppose you have a graph G with 9 vertices and 20 edges. What can the minimum degree $\delta(G)$ of this graph be?

A3. Since the sum of the degrees is $2 \cdot 20 = 40$, the *average* of the degrees is $\frac{40}{9} \approx 4.44$. So the minimum degree can be at most 4: if every vertex had degree 5 or more, then the sum of degrees would be at least $5 \cdot 9 = 45$. We can convince ourselves that it's possible to have a minimum degree of 0, 1, 2, 3, or 4.

We can generalize the answer to the first question. Let's say that a vertex v is **even** if $\deg(v)$ is an even number, and **odd** if $\deg(v)$ is an odd number. Then:

Corollary 1.2. *Every graph G must have an even number of odd vertices (possibly 0).*

Proof. By the handshake lemma, the sum of degrees is always an even number (twice the number of edges.) If you subtract of all the even degrees, you still have an even number. So the sum of all odd degrees is even.

The sum of an odd number of odd values is odd. The sum of an even number of odd values is even. So there must be an even number of odd degrees to sum, if we want an even total. □

2 Degrees and cycles

The following theorem will be very useful to us in a couple of weeks:

Theorem 2.1. *Let G be a graph whose minimum degree $\delta(G)$ is at least 2. Then G contains a cycle.*

The intuition for this theorem is as follows. Just start at any vertex of G and walk around, taking care not to leave a vertex the way you entered it. Eventually you will run into a vertex you've seen before. The first time that happens, your trajectory from that vertex and back forms a cycle.

Whenever we have an intuition of the form "keep doing this thing until it does what we want", this suggests a proof with the extremal principle. Just take (by the extremal principle) the situation in "this thing" has been done for as long as it can without "doing what we want". Then it is forced to "do what we want" in the very next step.

In this particular case, if we walk around for as long as we possibly can without revisiting a vertex, what we're getting is a very long path. This suggests the following proof:

Proof. Let $(v_1, v_2, \dots, v_k)$ be a longest path in G. (We know a longest path exists because there's an upper limit to the length of a path: the number of vertices in G.)

We know $\delta(G) \geq 2$ and therefore in particular $\deg(v_1) \geq 2$. Is it possible that v_1 has a neighbor w which is *not* one of $v_2, \dots, v_k$? It's not! In that case, $(w, v_1, v_2, \dots, v_k)$ would be a longer path.

So v_1 has at least two neighbors, which are all in the set $\{v_2, v_3, \dots, v_k\}$. One of v_1's neighbors is v_2: the next vertex on the path. This doesn't help us. But there must be another vertex v_i with $i > 2$ which is adjacent to v_1.

Then $(v_1, v_2, \dots, v_i, v_1)$ is the cycle we wanted. □

3 Average and minimum degree

Even if you have lots and lots and lots of edges, your minimum degree can be very small. For example, a 100-vertex graph might consist of a 99-vertex complete graph and a single isolated vertex. This has 4851 edges, which is close to the maximum number of edges a 100-vertex graph can have: 4950. And yet the minimum degree is 0, and we can't apply any nice results like Theorem 2.1 to this graph.

Here's a way to partially solve this problem:

Theorem 3.1. *Let G be a graph with average degree at least d. Then G contains a subgraph H with $\delta(H) > \frac{d}{2}$.*

This *could* be done using another application of the extremal principle: if you pick a subgraph H with the highest average degree, then it turns out to work. Rather than do this, we will use induction, so that we get to practice induction. (Take note of how we structure this proof to avoid the "induction trap"!)

Proof. We induct on n, the number of vertices in G.

When $n \leq d$, the theorem "holds trivially": that is, when $n \leq d$, the highest possible degree in G is $d-1$, so G cannot have average degree d or more to begin with! So there are no n-vertex graphs with average degree at least d, and all statements beginning "All n-vertex graphs with average degree at least d are..." are true. This could be our base case.

But if you're not happy with this, take $n = d+1$ as our base case. In this case, the highest possible degree in the graph is d. The average degree can only be this high if every vertex has degree d: if $G = K_{d+1}$. In this case, G itself is the subgraph H we're looking for. This base case also holds.

Either way, suppose that the theorem holds for all $(n-1)$-vertex graphs with average degree at least d. Let G be an n-vertex graph with average degree at least d.

At this point, let's say something about average degree. If G has vertices $v_1, v_2, \dots, v_n$, then the average of the degrees is

$$\frac{\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n)}{n} = \frac{2|E(G)|}{n}.$$

So the statement "G has average degree at least d" is equivalent to the statement "G has at least $\frac{1}{2}nd$ edges.

We're assuming G has at least $\frac{1}{2}nd$ edges. Also, if $\delta(G) > \frac{d}{2}$, then we are already done: we were looking for a subgraph with this minimum degree, and G itself can be that subgraph! So assume that G has a vertex v with $\deg(v) \leq \frac{d}{2}$.

In that case, let $G' = G - v$: the graph obtained from G by deleting v. We know that G' has $n-1$ vertices and at least $\frac{1}{2}nd - \frac{d}{2}$ edges: we started with at least $\frac{1}{2}nd$ edges, and we lost at most $\frac{d}{2}$ of them from deleting v. This simplifies to $\frac{1}{2}(n-1)d$, so G' has at least $\frac{1}{2}(n-1)d$ edges, which means G' has average degree at least d, too!

By applying the inductive hypothesis to G', we learn that G' has a subgraph H with $\delta(H) > \frac{d}{2}$. Since H is a subgraph of G', and G' is a subgraph of G, we have found the subgraph of G we wanted.

By induction, the theorem holds for graphs with any number of vertices. □

Corollary 3.2. *If G has n vertices and at least n edges, then G contains a cycle.*

Proof. If G has n vertices and at least n edges, it has average degree at least $d = 2$. By Theorem 3.1, G has a subgraph H with $\delta(H) > \frac{d}{2} = 1$. If $\delta(H) > 1$, then $\delta(H) \geq 2$, so by Theorem 2.1, H contains a cycle. This is also a cycle in G. □

4 Practice problems

1. Use Lemma 1.1 to find the number of edges in the cube graph Q_n without using induction.
2. In an earlier lecture, we gave an abstract definition of the Petersen graph: the vertices correspond to two-element subsets of $\{1, 2, 3, 4, 5\}$, and two vertices are adjacent if the subsets are disjoint.
   (a) Without going back to looking at a drawing of the Petersen graph, compute the number of edges it has directly from the definition.
   (b) Suppose that we generalize the graph: the vertices correspond to two-element subsets of $\{1, 2, \dots, n\}$, and two vertices are adjacent if the two subsets are disjoint.

   What is the degree of a vertex in this graph? How many edges does the graph have?
3. Suppose that G is a graph with 5 vertices and 7 edges. For which pairs (a, b) is it possible that $\delta(G) = a$ and $\Delta(G) = b$?

   For the cases where it is possible, give an example. For the cases where it is not possible, explain why not.
4. The five Platonic solids are:
   - The tetrahedron, which has 4 vertices and 4 triangular faces, with 3 faces meeting at every corner.
   - The cube (or hexahedron), which has 8 vertices and 6 square faces, with 3 faces meeting at every corner.
   - The octahedron, which has 6 vertices and 8 triangular faces, with 4 faces meeting at every corner.
   - The dodecahedron, which has 20 vertices and 12 pentagonal faces, with 3 faces meeting at every corner.
   - The icosahedron, which has 12 vertices and 20 triangular faces, with 5 faces meeting at every corner.

   In each case, find the number of edges where two faces meet. *(Later in the semester, we will explore further constraints on the Platonic solids.)*
5. Suppose G is a graph whose minimum degree $\delta(G)$ is at least 10. Then we can improve on the conclusion of Theorem 2.1: not only does G contain a cycle, but G contains a cycle of length at least 11.

   Prove this! The structure of the proof is very similar, except that instead of saying "So v_1 has at least two neighbors, ..." we can say "So v_1 has at least 10 neighbors, ..." and go from there. How can we use those neighbors to get a *long* cycle?
6. Adapt the proof idea of Lemma 1.1 to prove the following result: if G is a bipartite graph with bipartition (A, B), then
$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v).$$

7. (a) Suppose that G is a 10-vertex graph with $\delta(G) = 5$. Is it possible that G is not connected?

   If so, give an example. If not, explain why not.

   (b) Suppose that G is an 11-vertex graph with $\delta(G) = 6$. Is it possible that G is not connected?

   If so, give an example. If not, explain why not.

   (c) Generalize this argument: if G is an n-vertex graph, for what value of $\delta(G)$ can we be certain that G is connected? Prove your answer.

8. An open problem called *Conway's* 99*-graph problem* is to determine whether there is a 99-vertex graph with the following properties:

   - Every two adjacent vertices have exactly one common neighbor;
   - Every two non-adjacent vertices have exactly two common neighbors.

   If such a graph exists, then every vertex in it must have the same degree, d. What is d?