Math 3322: Graph Theory[1] Mikhail Lavrov

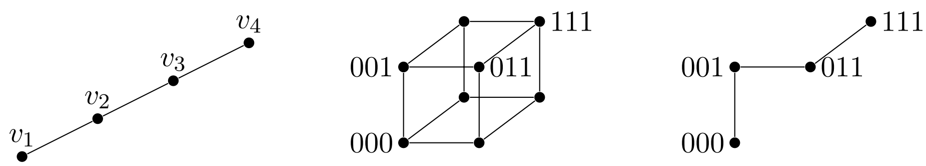
Lecture 9: Graph Isomorphisms

September 14, 2021 *Kennesaw State University*

1 Isomorphic graphs

Several times in this class, we have wanted to say that two graphs have the "same structure" even if they're not literally the same graph.

For example, our definition of a path graph P_n is a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edges $v_i v_{i+1}$ for $i = 1, 2, \dots, i+1$. But when we find the $000 - 111$ path $(000, 001, 011, 111)$ in the cube graph Q_3, we want to say that this path corresponds to "a P_4 subgraph" even though it is not literally P_4: its vertices are not named v_1, v_2, v_3, v_4.



Now we are going to define the formal way in which this subgraph is "the same as" P_4.

We define a **graph isomorphism** from graph G to graph H to be a function $f : V(G) \to V(H)$ with two properties:

1. f is a bijection—for every vertex $y \in V(H)$, there is exactly one vertex $x \in V(G)$ such that $f(x) = y$.
2. f preserves adjacency—for any two vertices $v, w \in V(G)$, v and w are adjacent in G if and only if $f(v)$ and $f(w)$ are adjacent in H.

Two graphs are **isomorphic** if there is an isomorphism from one graph to the other. In which direction? It doesn't matter: because we defined f to be a bijection in the definition above, it has an inverse $f^{-1} : V(H) \to V(G)$, and we can check that f^{-1} is an isomorphism from H to G.

The intuition is that isomorphic graphs are "the same graph, but with different vertex names". The graph isomorphism is a "dictionary" that translates between vertex names in G and vertex names in H.

In the diagram above, we can define a graph isomorphism from P_4 to the path subgraph of Q_3 by $f(v_1) = 000$, $f(v_2) = 001$, $f(v_3) = 011$, $f(v_4) = 111$. To check the second property of being an isomorphism, we verify that:

- v_1v_2, v_2v_3, and v_3v_4 are edges of P_4. Accordingly, 000 is adjacent to 001, 001 is adjacent to 011, and 011 is adjacent to 111 in the subgraph on the right.

[1]This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2021.php

- v_1v_3, v_1v_4, and v_2v_4 are **not** edges of P_4. Accordingly, 000 is not adjacent to 011 or 111, and 001 is not adjacent to 111, in the subgraph on the right.

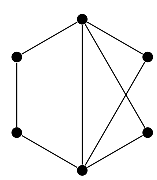
Essentially all the properties we care about in graph theory are preserved by isomorphism. For example, if G is isomorphic to H, then we can say that:

- G and H have the same number of vertices and edges.
- G is connected if and only if H is connected. More generally, G and H have the same number of connected components.
- G is bipartite if and only if H is bipartite.
- G and H have the same diameter.
- G and H have the same degree sequence; in particular, the same minimum degree and the same maximum degree. We can be more precise: if f is a graph isomorphism from G to H, and v is a vertex of G, then $\deg_G(v) = \deg_H(f(v))$.

Watch out for one common pitfall. Even if G and H share some vertices, the isomorphism between G and H does not have to care about the shared vertices—isomorphisms don't care about vertex names. If G and H both have a vertex v, and are isomorphic, v might do different things in the two graphs (for example, $\deg_G(v)$ might be different from $\deg_H(v)$).

However, in general, if you can describe some property without making reference to vertex names, then it should be preserved by isomorphism. For example, if G has the property "no two vertices of degree 4 in G are adjacent", and H is isomorphic to G, then H must also have this property.

This is very useful for proving that two graphs are *not* isomorphic. For example, how do we distinguish the complete bipartite graph $K_{2,4}$ from the graph below?



Some easy tests fail: $K_{2,4}$ has the same number of vertices (6) as the graph above, the same number of edges (8), and the same degree sequence $(4, 4, 2, 2, 2)$. However, either of the following tests distinguishes this graph from $K_{2,4}$:

- The graph above is not bipartite: for example, you can find two cycles of length 3 in it. $K_{2,4}$, of course, is bipartite.
- In $K_{2,4}$, the two vertices of degree 4 are not adjacent. In the graph above, they are.

2 Graph automorphisms (symmetries)

An **automorphism** of a graph G is an isomorphism from G to itself. The function $f : V(G) \to V(G)$ such that $f(v) = v$ for all vertices v is always an automorphism. However, there may be more complicated ones. For example, the path graph P_n has an automorphism that "reverses the path": $f(v_i) = v_{n+1-i}$.

Why are automorphisms useful? Because they let us describe ways in which a graph is "symmetric". This lets us avoid dealing with many identical cases when we're proving something about a graph.

Here's an example. First, we will prove a lemma about automorphisms of the cycle graph C_n, which admittedly takes a bit of work.

Lemma 2.1. *If x and y are any two adjacent vertices of the cycle graph C_n, then C_n has an automorphism f such that $f(x) = v_n$ and $f(y) = v_1$.*

Proof. There are several cases. I will just define the automorphism in each case, and leave you to check two facts: first, that f takes x to v_n and y to v_1; second, that f preserves adjacencies between vertices.

If $x = v_n$ and $y = v_1$ already, take the automorphism $f(v_i) = v_i$.

If $x = v_k$ and $y = v_{k+1}$, define the automorphism f by

$$f(v_i) = \begin{cases} v_{i+(n-k)} & i \leq k \\ v_{i-k} & i > k. \end{cases}$$

If $x = v_1$ and $y = v_n$, define f by $f(v_i) = v_{n+1-i}$.

Finally, if $x = v_k$ and $y = v_{k-1}$, we already know that there is an automorphism f_1 that takes x to v_1 and y to v_n (by the second case), and an automorphism f_2 that swaps v_1 and v_n (by the third case). Define f by $f(v_i) = f_2(f_1(v_i))$. □

But now we can use this lemma to make lots of proofs about C_n much easier, because we don't have to check many very similar cases! For example:

Theorem 2.1. *If xy is any edge of C_n, then $C_n - xy$ (the graph we obtain by deleting xy) is still connected.*

Proof. By Lemma 2.1, we may assume that $x = v_n$ and $y = v_1$. Here's how: let f be the automorphism that takes x to v_n and $y = v_1$. Then the same f is also an isomorphism between $C_n - xy$ and $C_n - v_n v_1$. So if we show that $C_n - v_n v_1$ is connected, we conclude that $C_n - xy$ is connected.

(Warning: you may see proofs that skip the explanation of how the automorphism helps us, and just say "because the graph has such-and-such symmetry, we may assume this-and-that".)

When $x = v_n$ and $y = v_1$, then $C_n - xy = P_n$, which we already know is connected. (We really did prove this at some point a few weeks ago.) So $C_n - xy$ is connected for all edges xy. □

3 How many r-regular graphs are there?

Disclaimer: we will not actually answer the question in the title of this section for most r. However, it has some easy answers when $r = 0, 1, 2$. By taking the complement, we get corresponding easy answers for $r = n-1, n-2, n-3$.

Claim 3.1. *Up to isomorphism, there is only one* 0*-regular graph on* n *vertices.*

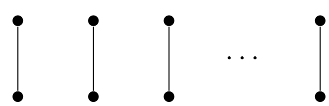
Proof. The words "up to isomorphism" here mean "any two such graphs are isomorphic". This claim really needs these words, because otherwise we would have to consider two 0-regular graphs on n vertices different if we just give the vertices different names.

Anyway, a 0-regular graph has 0 edges. So if we have two 0-regular graphs G and H on n vertices, any bijection $f : V(G) \to V(H)$ is an isomorphism. Two vertices are never adjacent in G, and they're also never adjacent in H, so f preserves adjacency. □

Up to isomorphism, there is also only one $(n-1)$-regular graph on n vertices: the complete graph K_n.

Claim 3.2. *Up to isomorphism, there is only one* 1*-regular graph on* n *vertices, when* n *is even. There are none when* n *is odd.*

We won't prove this claim because it's a bit tedious, but here is the only possible graph:



In particular, P_2 is the only connected 1-regular graph, on any number of vertices. In general, a $2k$-vertex 1-regular graph has k connected components, each isomorphic to P_2; we can define an isomorphism to the graph above by dealing with each component separately.

For 2-regular graphs, the story is more complicated. What we can say is:

Claim 3.3. *Up to isomorphism, there is only one* connected 2*-regular graph on* $n \geq 3$ *vertices: the cycle graph* C_n*.*

Proof. Let G be a connected 2-regular graph on n vertices. In particular, G has minimum degree 2, so by a result we proved earlier, we know that G has a cycle. Let $(x_1, x_2, x_3, \dots, x_k, x_1)$ be that cycle.

Each of the vertice $x_1, x_2, \dots, x_k$ has two neighbors on this cycle, but every vertex of G has degree 2 in total. Therefore these vertices have no other neighbors, and $\{x_1, x_2, \dots, x_k\}$ is a connected component of G.

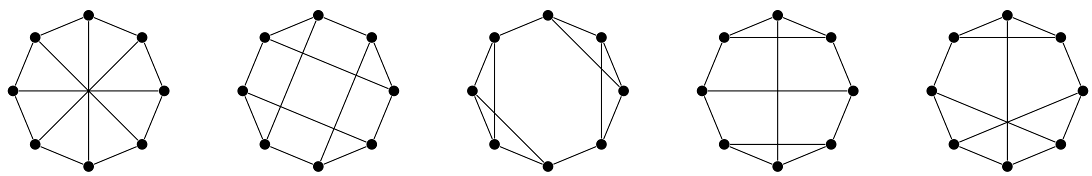
Since G is connected, this connected component must be all of G. Therefore $k = n$ and we can define an isomorphism f from G to C_n by $f(x_i) = v_i$ for $i = 1, \dots, n$. □

In general, there are many 2-regular graphs on n vertices; as many as there are ways to split up n into connected components of at least 3 vertices each.

For $r = 3$, already things become more complicated.

There is only one 3-regular graph on 4 vertices: the complete graph K_4. With 6 vertices, we get the complete bipartite graph $K_{3,3}$, and the so-called triangular prism graph: the graph you get if you take the vertices to be the corners of a triangular prism, and the edges to be the actual physical edges connecting them.

With 8 vertices, we get 6 different 3-regular graphs. One is the disjoint union of two copies of K_4. The other 5 are connected, and shown below.



Some notes:

- It is not exactly a *coincidence* that all 5 of these graphs have an 8-cycle, but don't expect this pattern to continue. It holds for 8-vertex graphs for some deeper reasons we won't cover.
- See if you can spot the cube graph Q_3: it must be one of these five graphs.
- It is a tricky, but good exercise to prove that none of these five graphs are isomorphic.

  (Hint: to distinguish most of them, look at the 3*-cycles.)*
- It is a frustrating and annoying exercise to prove that there are no other possible graphs. (Knowing that there must be an 8-cycle helps.)