Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 9: Graph isomorphisms

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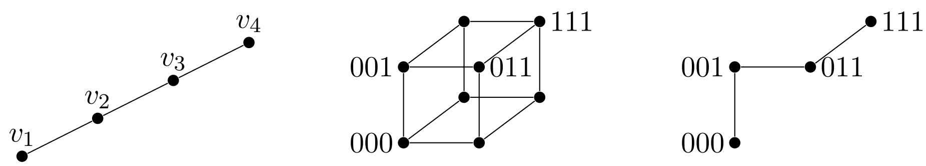
1 Isomorphic graphs

Several times in this class, we have wanted to say that two graphs have the "same structure" even if they're not literally the same graph.

For example, our definition of a path graph P_n is a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edges $v_i v_{i+1}$ for $i = 1, 2, \dots, i+1$. But when we find the $000 - 111$ path $(000, 001, 011, 111)$ in the cube graph Q_3, we want to say that this path corresponds to "a P_4 subgraph" even though it is not literally P_4: its vertices are not named v_1, v_2, v_3, v_4.



Now we are going to define the formal way in which this subgraph is "the same as" P_4.

We define a **graph isomorphism** from graph G to graph H to be a function $f \colon V(G) \to V(H)$ with two properties:

1. f is a bijection—for every vertex $y \in V(H)$, there is exactly one vertex $x \in V(G)$ such that $f(x) = y$.
2. f preserves adjacency—for any two vertices $v, w \in V(G)$, v and w are adjacent in G if and only if $f(v)$ and $f(w)$ are adjacent in H.

Two graphs are **isomorphic** if there is an isomorphism from one graph to the other. In which direction? It doesn't matter: because we defined f to be a bijection in the definition above, it has an inverse $f^{-1} \colon V(H) \to V(G)$, and we can check that f^{-1} is an isomorphism from H to G.

The intuition is that isomorphic graphs are "the same graph, but with different vertex names". The graph isomorphism is a "dictionary" that translates between vertex names in G and vertex names in H.

In the diagram above, we can define a graph isomorphism from P_4 to the path subgraph of Q_3 by $f(v_1) = 000$, $f(v_2) = 001$, $f(v_3) = 011$, $f(v_4) = 111$. To check the second property of being an isomorphism, we verify that:

- v_1v_2, v_2v_3, and v_3v_4 are edges of P_4. Accordingly, 000 is adjacent to 001, 001 is adjacent to 011, and 011 is adjacent to 111 in the subgraph on the right.

[1] This document comes from the Math 3322 course webpage: http://facultyweb.kennesaw.edu/mlavrov/courses/3322-fall-2023.php

- v_1v_3, v_1v_4, and v_2v_4 are **not** edges of P_4. Accordingly, 000 is not adjacent to 011 or 111, and 001 is not adjacent to 111, in the subgraph on the right.

Essentially all the properties we care about in graph theory are preserved by isomorphism. For example, if G is isomorphic to H, then we can say that:

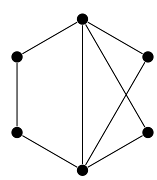
- G and H have the same number of vertices and edges.
- G is connected if and only if H is connected. More generally, G and H have the same number of connected components.
- G is bipartite if and only if H is bipartite.
- G and H have the same diameter.
- G and H have the same degree sequence; in particular, the same minimum degree and the same maximum degree. We can be more precise: if f is a graph isomorphism from G to H, and v is a vertex of G, then $\deg_G(v) = \deg_H(f(v))$.

Sometimes, we call these **invariants** of a graph: "invariant" because they not change when we rearrange the vertices.

Watch out for one common pitfall. Even if G and H share some vertices, the isomorphism between G and H does not have to care about the shared vertices—isomorphisms don't care about vertex names. If G and H both have a vertex v, and are isomorphic, v might do different things in the two graphs (for example, $\deg_G(v)$ might be different from $\deg_H(v)$).

However, in general, if you can describe some property without making reference to vertex names, then it should be preserved by isomorphism. For example, if G has the property "no two vertices of degree 4 in G are adjacent", and H is isomorphic to G, then H must also have this property.

This is very useful for proving that two graphs are *not* isomorphic. For example, how do we distinguish the complete bipartite graph $K_{2,4}$ from the graph below?



Some easy tests fail: $K_{2,4}$ has the same number of vertices (6) as the graph above, the same number of edges (8), and the same degree sequence $(4, 4, 2, 2, 2, 2)$. However, either of the following tests distinguishes this graph from $K_{2,4}$:

- The graph above is not bipartite: for example, you can find two cycles of length 3 in it. $K_{2,4}$, of course, is bipartite.
- In $K_{2,4}$, the two vertices of degree 4 are not adjacent. In the graph above, they are.

2 Graph automorphisms (symmetries)

An **automorphism** of a graph G is an isomorphism from G to itself. The function $f\colon V(G) \to V(G)$ such that $f(v) = v$ for all vertices v is always an automorphism. However, there may be more complicated ones. For example, the path graph P_n has an automorphism that "reverses the path": $f(v_i) = v_{n+1-i}$.

Why are automorphisms useful? Because they let us describe ways in which a graph is "symmetric". This lets us avoid dealing with many identical cases in proofs about that graph.

Here's an example. First, we will prove a lemma about automorphisms of the cycle graph C_n, which admittedly takes a bit of work.

Lemma 2.1. *If x and y are any two adjacent vertices of the cycle graph C_n, then C_n has an automorphism f such that $f(x) = v_1$ and $f(y) = v_n$.*

Proof. If g and h are two automorphisms of C_n, then their composition $g \circ h$ is also an automorphism. So we will build the automorphism f we want by composing simpler automorphisms.

One simple automorphism of C_n is the "left shift" automorphism s, defined by

$$s(v_i) = \begin{cases} v_{i-1} & \text{if } i > 1, \\ v_n & \text{if } i = 1. \end{cases}$$

If it happens that $x = v_k$ and $y = v_{k-1}$ for some $k \geq 2$, then applying the left shift automorphism $k-1$ times will take x to v_1 and y to v_n. (Also, if $x = v_1$ and $y = v_n$ already, then the identity automorphism is the automorphism we're looking for.)

However, x and y might appear in the other order around the cycle. In this case, we'll need another simple automorphism of C_n: the "reverse" automorphism r, defined by $r(v_i) = v_{n+1-i}$ for all i.

This is the automorphism we're looking for if $x = v_n$ and $y = v_1$. Finally, if $x = v_{k-1}$ and $y = v_k$ for some $k \geq 2$, then we can apply the left shift automorphism $k-1$ times (taking x to v_n and y to v_1) then apply the reverse automorphism. □

Now we can use this lemma to make lots of proofs about C_n much easier, because we don't have to check many very similar cases! For example:

Theorem 2.2. *If xy is any edge of C_n, then $C_n - xy$ (the graph we obtain by deleting xy) is still connected.*

Proof. By Lemma 2.1, we may assume that $x = v_1$ and $y = v_n$. Here's how: let f be the automorphism that takes x to v_1 and $y = v_n$. Then the same f is also an isomorphism between $C_n - xy$ and $C_n - v_1v_n$. So if we show that $C_n - v_1v_n$ is connected, we conclude that $C_n - xy$ is connected.

(Warning: you may see proofs that skip the explanation of how the automorphism helps us, and just say "because the graph has such-and-such symmetry, we may assume this-and-that".)

When $x = v_1$ and $y = v_n$, then $C_n - xy = P_n$, which we already know is connected. (We really did prove this at some point a few weeks ago.) So $C_n - xy$ is connected for all edges xy. □

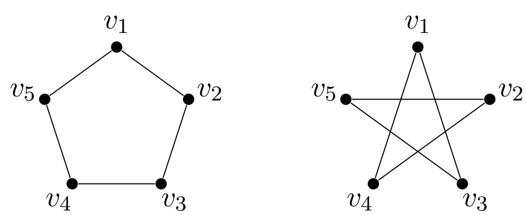
3 Self-complementary graphs

This section of the notes is a look at an interesting self-contained puzzle about graphs, which we now have the language to explore.

Define a graph G to be **self-complementary** if G is isomorphic to $\overline{G}$, the complement of G. For example, the path graph P_4 is self-complementary:

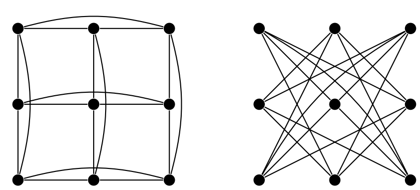
One possible isomorphism from the graph on the left to the graph on the right is given by $f(v_1) = v_2$, $f(v_2) = v_4$, $f(v_3) = v_1$, $f(v_4) = v_3$.

If you look for self-complementary graphs on 5 vertices, you'll probably find the 5-cycle:



There are many isomorphisms here (since C_5 has many automorphisms). One of them is given by $f(v_1) = v_1$, $f(v_2) = v_3$, $f(v_3) = v_5$, $f(v_4) = v_2$, and $f(v_3) = v_4$.

A more complicated example is a graph on 9 vertices known as the 3×3 "rook graph". Here, if we imagine the vertices as being laid out in a 3×3 grid, we declare vertices to be adjacent if they are in the same row or the same column. (The name "rook graph" comes from the way that a rook moves in chess.) In the diagram below, the rook graph as we just defined it is shown on the left, and its complement is shown on the right:



Can you find an isomorphism between these two graphs? *(Hint: you can start by declaring that the center vertex on the left maps to the center vertex on the right. From there, a lot of your decisions will be forced.)*

We can say a lot about the structure of a self-complementary graph. For example, if G is an n-vertex graph with m edges, then $\overline{G}$ has $\binom{n}{2} - m$ edges. If we want G to be isomorphic to $\overline{G}$, then we want $m = \binom{n}{2} - m$, which means $m = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$.

What's more, if G is self-complementary, its degree sequence must be symmetric. Suppose G has a vertex of degree k; then that same vertex has degree $n-1-k$ in $\overline{G}$. This means there must also be a vertex of degree $n-1-k$ in G: in fact, as many of these vertices as there are vertices of degree k. We see this in P_4, which has degree sequence $2, 2, 1, 1$. The other two examples we have take the easy way out by being regular of degree $\frac{n-1}{2}$. But from the degree sequence point of view, it's possible that another 5-vertex self-complementary graph exists, with degree sequence $3, 3, 2, 1, 1$. (In fact, there is such a graph; can you find it?)

When we come up with a new definition, and start playing around with it, there is the danger of putting in a lot of work for nothing. What if we spend hours and hours proving theorems about self-complementary graphs, only to find out that actually, we've found all the ones that exist, and there are no more? This would be very disappointing, so maybe it's best to first ask: are there actually many self-complementary graphs?

From what we've done, we can actually prove a limitation on their existence.

Proposition 3.1. *A self-complementary graph on n vertices can only exist if n has the form $4k$ or $4k+1$ for some integer k.*

Proof. We know that a self-complementary graph on n vertices must have $\frac{n(n-1)}{4}$ edges, so in particular, $\frac{n(n-1)}{4}$ must be an integer. One of n or $n-1$ is always odd, so the other one must be divisible by 4. If n is divisible by 4, then $n = 4k$ for some integer k; if $n-1$ is divisible by 4, then $n = 4k+1$ for some integer k. □

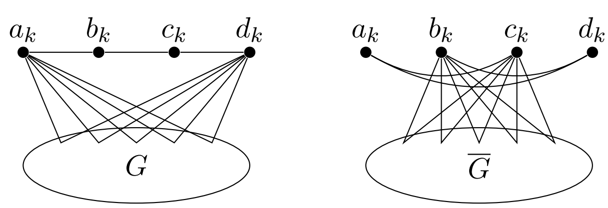
However, beyond that, there are no restrictions! Here's the formal statement.

Theorem 3.2. *For all integers $k \geq 1$, there are self-complementary graphs on $4k$ and $4k+1$ vertices.*

Proof. We induct on k. Our base cases for $k = 1$ are the path graph P_4 and the cycle graph C_5, which we've already seen.

Now assume that a self-complementary graph G on either $4k$ or $4k+1$ vertices exists; the same argument will actually apply to both cases. We'll use it to build a larger self-complementary graph with 4 more vertices (either $4(k+1)$ or $4(k+1)+1$ vertices).

Here's how: take G, and add 4 new vertices a_k, b_k, c_k, d_k. Between them, add edges a_kb_k, b_kc_k, and c_kd_k. Also, join every vertex in G by an edge to a_k and d_k (but *not* b_k or c_k).

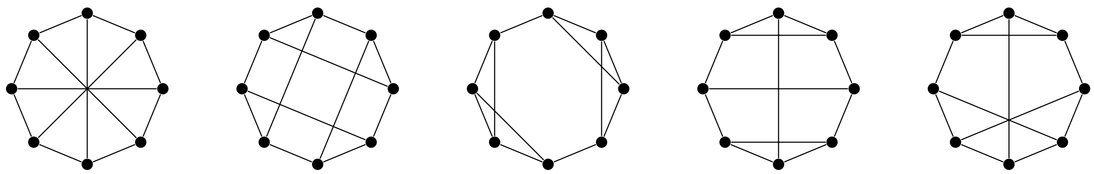


If $f\colon V(G) \to V(G)$ is an isomorphism from G to $\overline{G}$, extend it to the four new vertices by defining $f(a_k) = c_k$, $f(b_k) = a_k$, $f(c_k) = d_k$, and $f(d_k) = b_k$. You can check (looking at the diagram above) that this defines an isomorphism from the bigger graph to its complement, showing that it's also self-complementary.

By induction, we can generate self-complementary graphs of every possible size! □

4 Practice problems

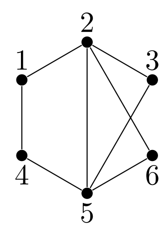
1. Prove that none of these five graphs are isomorphic: find invariants distinguishing them all from each other.



2. In fact, the five graphs shown above are the complete collection of 3-regular connected graphs on 8 vertices, up to isomorphism.

   In particular, this means that the cube graph Q_3 must be isomorphic to one of the five graphs above. Which one is it isomorphic to?

3. Find all automorphisms of the graph shown below: that is, all functions $f\colon \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$ that preserve the edges of the graph.



   (Hint: there are four, and one of them is very boring.)

4. Let G and H be isomorphic graphs. Prove the following:

   (a) G is connected if and only if H is connected.

   (b) G is bipartite if and only if H is bipartite.

   (c) G and H have the same number of vertices.

   (d) G and H have the same number of edges.

5. Suppose G and H are two graphs with the same vertex set: $V(G) = V(H)$. In the previous lecture, we talked about what happens if G and H have the same vertex degrees: $\deg_G(v) = \deg_H(v)$ for all vertices v.

   Prove that this neither implies nor is implied by G being isomorphic to H. In other words:

   (a) Give an example where $\deg_G(v) = \deg_H(v)$ for all vertices v, but G and H are not isomorphic.

   (b) Give an example where G and H are isomorphic, but $\deg_G(v)$ is not equal to $\deg_H(v)$ for at least some vertices v.

6. For $n \geq 5$, the walk (v_1, v_3, v_n) is a $v_1 - v_n$ walk in the complement $\overline{C}_n$.

   Use this fact and Lemma 2.1 to write a short proof that when $n \geq 5$, $\overline{C}_n$ is connected.