Math 3272: Linear Programming[1] Mikhail Lavrov

In-class problems for Lecture 3

August 23, 2022 *Kennesaw State University*

1 Problems to work on first

1. The system of equations below has infinitely many solutions. Solve for y and z in terms of x.

$$\begin{cases} 3x + 2y - 3z = -1 \\ 3x - y + 2z = 2 \end{cases}$$

2. The following system of equations has already been solved for x_1, x_2, x_3 in terms of x_4, x_5:

$$\begin{cases} x_1 + 3x_2 - 2x_3 + x_4 - x_5 = 1 \\ -2x_1 + x_2 + 2x_4 - x_5 = 1 \\ x_1 + x_2 - x_3 - x_4 = -1 \end{cases} \quad \rightsquigarrow \quad \begin{cases} x_1 = 2 - x_4 \\ x_2 = 5 - 4x_4 + x_5 \\ x_3 = 8 - 6x_4 + x_5 \end{cases}$$

   (a) Find two different particular solutions $(x_1, x_2, x_3, x_4, x_5)$ to this system of equations.

   (b) Solve for x_1, x_2, x_5 in terms of x_3, x_4 instead. Try to do as little additional work as possible.

3. Consider the following system of equations:

$$\begin{cases} 3x_1 + 5x_2 + x_3 - 2x_4 = 4 \\ x_1 + 2x_2 + x_3 - x_4 = -1 \end{cases}$$

   (a) Write this system of equations in matrix form: as $A\mathbf{x} = \mathbf{b}$, where A is a 2×4 matrix, $\mathbf{x}$ is the column vector of our variables $x_1, \dots, x_4$, and $\mathbf{b}$ is a 2×1 column vector.

   (b) Take the first two columns of A only. Find the inverse of this 2×2 matrix.

   (c) Left-multiply both sides of the matrix equation $A\mathbf{x} = \mathbf{b}$ by the inverse matrix you've found.

   (d) Your result should now be row-reduced. Use it to solve for x_1, x_2 in terms of x_3, x_4.

4. Consider the following system of equations, already written in matrix form:

$$\begin{bmatrix} 2 & 1 & -5 \\ 0 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

   (a) Left-multiply both sides of this matrix equation by the row vector $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$.

   (b) What does the result tell you about the system of equations?

[1] This document comes from the Math 3272 course webpage: https://facultyweb.kennesaw.edu/mlavrov/courses/3272-fall-2022.php

2 Challenge problems

5. Describe all solutions $(x_1, x_2, \ldots, x_n)$ to the system of equations below.

$$\begin{cases} x_1 & - 2x_2 & + x_3 = 0 \\ x_2 & - 2x_2 & + x_3 = 0 \\ x_3 & - 2x_2 & + x_3 = 0 \\ & \vdots & = \vdots \\ x_{n-2} & - 2x_{n-1} & + x_n = 0 \end{cases}$$

6. Take a look at the system of equations in problem 2 again. (It is especially useful to look at the given solution for x_1, x_2, x_3 in terms of x_4, x_5.)

   Suppose we are only considering nonnegative solutions to the system: solutions with

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

   In that case, answer the following questions; try to give a reason why in each case.

   (a) Is there a nonnegative solution where $x_4 = x_5 = 0$?

   (b) Is there a nonnegative solution where $x_1 > 2$?

   (c) Is there a nonnegative solution where $x_2 > 5$?

   (d) Among all nonnegative solutions, what is the highest possible value of x_4?

   (e) Among all nonnegative solutions where $x_5 = 0$, what is the highest possible value of x_4?

7. Consider the following linear program:

$$\begin{array}{rl} \underset{x,y\in\mathbb{R}}{\text{maximize}} & x + y \\ \text{subject to} & 2x + 3y \leq 15 \\ & x + 2y \leq 9 \\ & 2x + y \leq 12 \\ & x, y \geq 0 \end{array}$$

   (a) Without trying to solve the linear program, can you give a convincing argument for why there is no feasible solution (x, y) where $x + y$ is 10 or higher?

   (b) An shadowy figure cryptically tells you "take the sum of the first two inequalities, then divide by three".

   How can this help you get a better upper bound on $x + y$ than what you got in part (a)?

   (c) Can you find an even better upper bound on $x + y$ in the same way as in part (b)?