Math 3272: Linear Programming[1] Mikhail Lavrov

Lecture 6: Several examples of the simplex method

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1 An example with nothing weird going on

Today, we will skip the word problem and go straight to the linear program.

Problem 1. *Solve the following linear program (put in equational form on the right).*

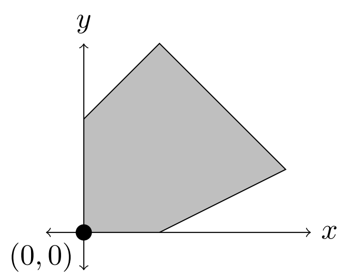
$$\begin{array}{ll} \underset{x,y\in\mathbb{R}}{\text{maximize}} & 2x+3y \\ \text{subject to} & -x+\ \ y \le 3 \\ & x-2y \le 2 \\ & x+\ \ y \le 7 \\ & x,y \ \ge 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \underset{x,y,w_1,w_2,w_3\in\mathbb{R}}{\text{maximize}} & 2x+3y \\ \text{subject to} & -x+\ \ y+w_1 = 3 \\ & x-2y+w_2 = 2 \\ & x+\ \ y+w_3 = 7 \\ & x,y,w_1,w_2,w_3 \ge 0 \end{array}$$

Adding slack variables has a convenient bonus effect. The slack variables (w_1, w_2, w_3) form a convenient set of basic variables to start with, for two reasons:

- The dictionary will already be row-reduced for the slack variables, since each one shows up in only one equation. This will be true any time we add slack variables.
- The basic solution is $(x, y, w_1, w_2, w_3) = (0, 0, 3, 2, 7)$, which is feasible. This happens whenever our starting inequalities are all upper bounds with a positive constant on the right-hand side. So it's not always useful, but sometimes makes our lives easier.

Here is our starting dictionary, and a graph of the feasible region (of the original linear program in x and y) with the corresponding basic feasible solution marked:

$$\begin{array}{r} \max \zeta = 0 + 2x + 3y \\ \hline w_1 = 3 + \ \ x - \ \ y \\ w_2 = 2 - \ \ x + 2y \\ w_3 = 7 - \ \ x - \ \ y \end{array}$$



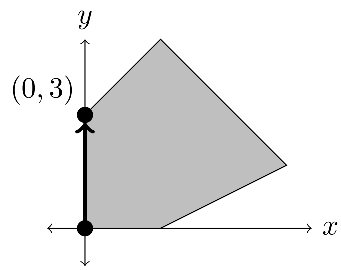
In a well-behaved 2-dimensional linear program, exactly 2 constraints should be tight at each corner. Specifically, it's the ones corresponding to the nonbasic variables.

In this dictionary, the nonbasic variables are x and y, and the constraints they "own" are the $x \ge 0$ and $y \ge 0$ constraints. So we should be at the corner where these two constraints are tight: the intersection of $x = 0$ and $y = 0$.

[1]This document comes from the Math 3272 course webpage: https://facultyweb.kennesaw.edu/mlavrov/courses/3272-fall-2022.php

Let's bring y into the basis. (This is an arbitrary choice: we could also have chosen x.) Since w_2's coefficient of y is 2, it's not a valid leaving variable; w_1 and w_3 have ratios of $\frac{3}{1}$ and $\frac{7}{1}$, of which the smallest is 3. So y replaces w_1 in the basis, giving us the new dictionary below:

$$\begin{array}{rl} \max \zeta = & 9 + 5x - 3w_1 \\ \hline y = & 3 + x - w_1 \\ w_2 = & 8 + x - 2w_1 \\ w_3 = & 4 - 2x + w_1 \end{array}$$



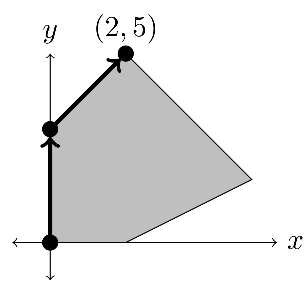
The basic feasible solution is $(x, y, w_1, w_2, w_3) = (0, 3, 0, 8, 4)$. The nonbasic variables of the new dictionary are x and w_1. As before, x "owns" the $x \geq 0$ constraint. Meanwhile, w_1 "owns" the $w_1 \geq 0$ constraint, but in the original linear program, this was the $-x+y \leq 3$ constraint. We should be at the corner where $x = 0$ and $-x + y = 3$ meet, and indeed, these lines meet at $(0, 3)$.

The choice of entering variable corresponds to picking the direction in which we went around the polygon: which edge out of $(0, 0)$ we used. The edge from $(0, 0)$ to $(0, 3)$ moves away from the $y \geq 0$ constraint, so y is the variable that becomes basic. We could also have brought x into the basis, moving away from the $x \geq 0$ constraint.

But now, at $(0, 3)$, there is only one good choice of entering variable. We don't want to go back to $(0, 0)$, so the only choice is to continue going clockwise. In the dictionary, this corresponds to how we don't want to bring w_1 back into the basis (its reduced cost is negative, so this would decrease ζ). Instead, the only helpful entering variable is x, whose reduced cost is positive.

In x's column, the coefficients of y and w_2 are both positive, so those can't be leaving variables. Therefore x replaces w_3 in the basis, giving us the new dictionary below:

$$\begin{array}{rl} \max \zeta = & 19 - \frac{1}{2}w_1 - \frac{5}{2}w_3 \\ \hline y = & 5 - \frac{1}{2}w_1 - \frac{1}{2}w_3 \\ w_2 = & 10 - \frac{3}{2}w_1 - \frac{1}{2}w_3 \\ x = & 2 + \frac{1}{2}w_1 - \frac{1}{2}w_3 \end{array}$$



In this dictionary, all reduced costs are negative. Therefore ζ is maximized and $(x, y) = (2, 5)$ is the optimal solution.

(Here, w_1 and w_3 are nonbasic. The constraints they "own" are the $-x + y \leq 3$ constraint and the $x + y \leq 7$ constraint. So we end up at the corner point where the lines $-x + y = 3$ and $x + y = 7$ intersect.)

If we had decided pivot around x first, rather than y, we would have arrived at the same final answer, but going counterclockwise around the feasible region instead. There would have been three steps, not two, because there are three edges to take when going around that way.

2 An unbounded linear program

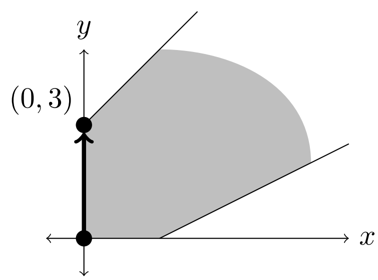
Let's modify Problem 1, removing the constraint $x + y \leq 7$ (corresponding to the slack variable w_3:

Problem 2. *Solve the following linear program (put in equational form on the right).*

$$\begin{array}{ll} \underset{x,y\in\mathbb{R}}{\textit{maximize}} & 2x+3y \\ \textit{subject to} & -x+\ \ y \leq 3 \\ & \ \ \ x-2y \leq 2 \\ & \ \ \ \ x,y \ \geq 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \underset{x,y,w_1,w_2\in\mathbb{R}}{\textit{maximize}} & 2x+3y \\ \textit{subject to} & -x+\ \ y+w_1 \qquad = 3 \\ & \ \ \ x-2y \qquad +w_2 = 2 \\ & \qquad x,y,w_1,w_2 \ \geq 0 \end{array}$$

Our first iteration of the simplex method will be nearly the same with Problem 2 as it was with Problem 1, and will also bring us to the point $(0, 3)$. We can quickly get the dictionary for that point by dropping the equation for w_3:

$$\begin{array}{r} \max \zeta = 9 + 5x - 3w_1 \\ \hline y = 3 + \ \ x - \ \ w_1 \\ w_2 = 8 + \ \ x - 2w_1 \end{array}$$



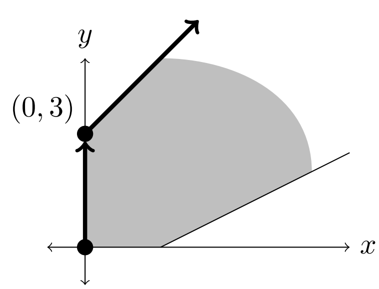
Looking at the diagram, we see what's about to happen: the feasible region is unbounded in the direction we want to go.

It's still a good idea to bring x into the basis: it still has a positive reduced cost. But now, both basic variables are ruled out at the first stage: both of them have a positive coefficient in x's column, so neither of them decreases as x increases. There is no leaving variable to choose.

This is what it looks like when the linear program is unbounded, and we can improve the objective value as much as we want. There is no optimal solution.

From this dictionary, we can also learn a bit about *how* the linear program is unbounded. The variables doing useful work are the basic variables y, w_2 and the entering variable x. All other variables (just w_1 in this case) should be set to 0. Setting w_1 to 0 and getting rid of it in every other row gives us the following pseudo-dictionary:

$$\begin{array}{r} \max \zeta = 9 + 5x \\ \hline y = 3 + \ \ x \\ w_2 = 8 + \ \ x \\ w_1 = 0 \quad \end{array}$$



We get better and better solutions as we travel along the line $y = 3 + x$, increasing x as much as we want: the objective value increases as $\zeta = 9 + 5x$. All our variables remain positive (including the slack variables w_1 and w_2), so the solution remains feasible the whole way.

All this is happening behind the scenes when we do any pivot step. But here, because the coefficients in x's column were both positive, the slopes of $y = x + 3$ and $w_2 = x + 8$ are both positive, which means that we can increase x without a limit. And since x had a positive reduced cost of 5, we know that this gives us arbitrarily large objective values.

Whenever we learn from the dictionary that the linear program is unbounded, we can perform such an analysis to find an infinite ray of feasible solutions along which the objective value improves without bound.

3 An example of degenerate pivoting

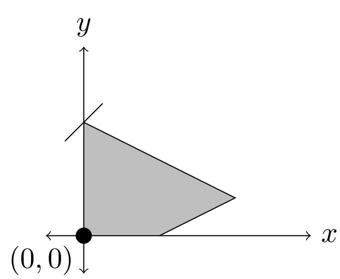
For our final problem, let's replace the $x + y \leq 7$ constraint by the constraint $x + 2y \leq 6$:

Problem 3. *Solve the following linear program (put in equational form on the right).*

$$\begin{array}{rrl} \underset{x,y\in\mathbb{R}}{\textit{maximize}} & 2x + 3y & \\ \textit{subject to} & -x + \ \ y & \leq 3 \\ & x - 2y & \leq 2 \\ & x + 2y & \leq 6 \\ & x, y & \geq 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{rrrrrl} \underset{x,y,w_1,w_2,w_3\in\mathbb{R}}{\textit{maximize}} & 2x + 3y & & & & \\ \textit{subject to} & -x + \ \ y & + w_1 & & & = 3 \\ & x - 2y & & + w_2 & & = 2 \\ & x + 2y & & & + w_3 & = 6 \\ & & & & x, y, w_1, w_2, w_3 & \geq 0 \end{array}$$

To see what makes Problem 3 different from Problem 1, let's take a look at the initial dictionary and especially at the feasible region:

$$\begin{array}{rcl} \max \zeta & = & 0 + 2x + 3y \\ \hline w_1 & = & 3 + \ \ x - \ \ y \\ w_2 & = & 2 - \ \ x + 2y \\ w_3 & = & 6 - \ \ x - 2y \end{array}$$

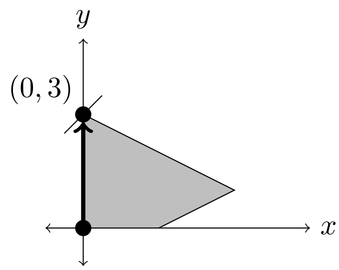


The constraint $-x + y \leq 3$ is just barely irrelevant. The line $-x + y = 3$ touches the feasible region at the corner point $(3, 0)$, and doesn't change the feasibility of anything. How will things change? Let's find out!

We can proceed as before and bring y into the basis. As before, w_2 is out of consideration because it has a positive coefficient in y's column. Meanwhile, w_1 and w_3 have ratios $\frac{3}{1}$ and $\frac{6}{2}$, so they are tied for having the smallest ratio. As we increase y, w_1 and w_3 will decrease and hit 0 at the same time.

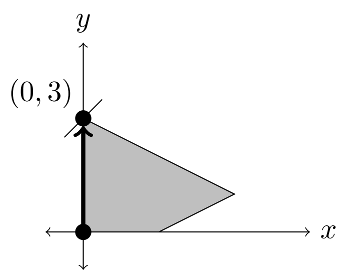
Which leaving variable do we choose in such a case? Right now, we have no way to tell which choice is better, but either choice will give us a feasible dictionary. Let's choose w_1, because that's what we did last time. We get:

$$\begin{array}{r}
\max \zeta = 9 + 5x - 3w_1 \\
\hline
y = 3 + \ \ x - \ \ w_1 \\
w_2 = 8 + \ \ x - 2w_1 \\
w_3 = 0 - 3x + 2w_1
\end{array}$$



Here is where things start to go wrong. Our next entering variable must be x, because it's the only variable with positive reduced cost. As x increases, y and w_2 also increase, so they will not leave the basis: the leaving variable must be w_3. But when we make this happen, the dictionary changes, but the values of all the variables stay the same!

$$\begin{array}{r}
\max \zeta = 9 + \frac{1}{3}w_1 - \frac{5}{3}w_3 \\
\hline
y = 3 - \frac{1}{3}w_1 - \frac{1}{3}w_3 \\
w_2 = 8 - \frac{4}{3}w_1 - \frac{1}{3}w_3 \\
x = 0 + \frac{2}{3}w_1 - \frac{1}{3}w_3
\end{array}$$



The problem is that the three lines $x = 0$, $-x + y = 3$, and $x + 2y = 6$ all meet at the point $(0, 3)$. Previously, when x and w_1 were nonbasic, we thought of $(0, 3)$ as the intersection of $x = 0$ and $-x + y = 3$. Now, w_1 and w_3 are nonbasic, and we've "moved" to the intersection of $-x + y = 3$ and $x + 2y = 6$. This is also $(0, 3)$.

This is called **degenerate pivoting**. There's one big problem with degenerate pivoting:

- Usually, we can say, "The simplex method is always improving the value of ζ, so it can never revisit the same corner point. Since there's only finitely many corner points, it *has* to reach the right one eventually."
- With degenerate pivoting, the value of ζ does not always improve. So we have no guarantee that the simplex method won't keep going forever, stuck at the same corner point being represented in different ways.

In this example, we'll leave the point $(0, 3)$ after one more step. But in more complicated examples, when many constraints meet at one poit, staying at that point forever is a real danger. To avoid this, we'll need to develop **pivoting rules** that avoid infinite loops, by telling us the right variable to remove from the basis in cases when there's a tie.