Lecture 7: Two-phase simplex methods

September 6, 2022 *Kennesaw State University*

1 The problem with initialization

1.1 A tricky example

Consider the following problem:

Problem 1. *You have an important assignment due in 5 hours. You're working on it in a coffee shop, and so you're trying to bribe yourself to work on it by a combination of fancy coffee and sweet tea.*

You'll need at least one cup an hour to keep you focused. To stay awake until the assignment is due, you'll need at least 7 "units" of caffeine (if we say a unit of caffeine is the amount in a cup of tea; there's 3 unit in a cup of coffee). Finally, to have the energy to work on the assignment, you need at least 6 units of sugar (the amount in a cup of coffee; a cup of sweet tea has 2 units).

If every cup of coffee costs \$4.50 and every cup of tea costs \$3, what is the cheapest way to make this work?

We can write this problem as follows:

$$\begin{array}{ll} \underset{x_1,x_2\in\mathbb{R}}{\text{minimize}} & 4.5x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \ge 5 \\ & 3x_1 + x_2 \ge 7 \\ & x_1 + 2x_2 \ge 6 \\ & x_1, x_2 \ge 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \underset{x_1,x_2,w_1,w_2,w_3\in\mathbb{R}}{\text{minimize}} & 4.5x_1 + 3x_2 \\ \text{subject to} & -x_1 - x_2 + w_1 = -5 \\ & -3x_1 - x_2 + w_2 = -7 \\ & -x_1 - 2x_2 + w_3 = -6 \\ & x_1, x_2, w_1, w_2, w_3 \ge 0 \end{array}$$

The difficulty in adapting our methods to this problem is this: how do we find an initial basic feasible solution?

1.2 The problem in general

In the previous lecture, we relied on a shortcut for starting the simplex method: we assumed that the point $\mathbf{x} = \mathbf{0}$ is feasible for our linear program. This naturally happens, for example, in production problems where all our constraints are resource constraints: they put upper bounds on how big $x_1, x_2, \dots, x_n$ get, but no lower bounds.

However, not all linear programs look like this! In particular, in the linear program above, none of our constraints are satisfied when $\mathbf{x} = \mathbf{0}$, except for the nonnegativity constraints.

We'll distinguish between two cases where starting with $\mathbf{x} = \mathbf{0}$ does not make sense:

[1] This document comes from the Math 3272 course webpage: https://facultyweb.kennesaw.edu/mlavrov/courses/3272-fall-2022.php

- (Easier case) All our constraints are inequality constraints, which we'll have to add slack variables to before getting a problem in equational form.
- (Harder case) We are dealing with a problem that's already in equational form ($A\mathbf{x} = \mathbf{b}$ where $\mathbf{x} \geq \mathbf{0}$), possibly because we started out with some equational constraints.

In both cases, the solution is the **two-phase simplex method**. In this method, we:

1. Solve an auxiliary problem, which has a built-in starting point, to determine if the original linear program is feasible. If we succeed, we find a basic feasible solution to the original linear program.
2. From that basic feasible solution, solve the linear program the way we've done it before.

The auxiliary problem will be easier to state, and have fewer additional variables, in the case we're calling the "easier cases"—such as the example problem. So we will begin there.

2 The two-phase method for inequalities

We adjust our problem by adding a new **artificial variable** x_0 that lets us violate each constraint by some amount. When we replace an $a \leq b$ constraint by $a - x_0 \leq b$, this lets us violate the original constraint by a margin of up to x_0. No matter what our other variables are set to, if we make x_0 large enough, we can satisfy every equation.

Of course, we don't want to make x_0 large, because we're not interested in solutions that violate all our constraints. So our first step will be to minimize an **auxiliary objective function**: we will minimize $\xi = x_0$. If we can get x_0 down to 0, then we're not violating any constraints, and so we have a feasible solution for our original problem!

(Note: this is a slight departure from how the problem is described in the textbook; the textbook has not yet introduced minimization problems, and so it describes everything in terms of maximizing $-\xi$. Also, feel free to use a different variable instead of ξ (xi) if you have trouble writing ξ.)

This is called the **phase one problem**. Here is the full description, in our original example:

$$\begin{array}{rrcl} \underset{x_0,x_1,x_2,w_1,w_2,w_3 \in \mathbb{R}}{\text{minimize}} & & x_0 & \\ \text{subject to} & -x_1 - x_2 + w_1 - x_0 & = & -5 \\ & -3x_1 - x_2 + w_2 - x_0 & = & -7 \\ & -x_1 - 2x_2 + w_3 - x_0 & = & -6 \\ & x_0, x_1, x_2, w_1, w_2, w_3 & \geq & 0 \end{array}$$

As usual, we'll set $x_1 = x_2 = 0$ in our initial feasible solution. We'll need to set $x_0 = 7$, because that's the largest number on the right-hand side. Then $w_2 = 0$ satisfies our second equation, and we can set $w_1 = 7 - 5 = 2$ and $w_3 = 6 - 5 = 1$ to satisfy the second and third equation.

That's an unsystematic description of how we get our initial dictionary, though. We begin by taking our basic variables to be w_1, w_2, w_3:

$$\begin{array}{rl} \xi = & 0 \qquad\qquad\qquad + x_0 \\ \hline w_1 = & -5 + \ \ x_1 + \ \ x_2 + x_0 \\ w_2 = & -7 + 3x_1 + \ \ x_2 + x_0 \\ w_3 = & -6 + \ \ x_1 + 2x_2 + x_0 \end{array}$$

This is not feasible: all three of w_1, w_2, w_3 are negative in the basic solution. Our first step in the phase one problem is always, ignoring any pivoting rules, to bring x_0 into the basis, and take w_2 (the variable with the most negative value) out of the basis. This is guaranteed to lead us to a feasible dictionary:

$$\begin{array}{rl} \xi = & 7 - 3x_1 - x_2 + w_2 \\ \hline w_1 = & 2 - 2x_1 \qquad\ + w_2 \\ x_0 = & 7 - 3x_1 - x_2 + w_2 \\ w_3 = & 1 - 2x_1 + x_2 + w_2 \end{array}$$

(It will always be the case that the equation for ξ at the top matches the equation for x_0, for as long as x_0 is a basic variable. I will keep writing the same equation in both places, just to match the usual way we write a dictionary.)

Now we can proceed to solve this linear program in the usual way. Since we're minimizing ξ, we should pivot on entries that have a negative reduced cost. In this example, pivoting on x_2 turns out to be the best choice. The only possible leaving variable is x_0 (since it is the only one with a negative coefficient on x_2. Solving x_0's equation for x_2 gives $x_2 = 7 - 3x_1 + w_2 - x_0$, and then we will just substitute that in for x_2 in the other equations:

$$\begin{array}{rl} \xi = & 7 - 3x_1 + w_2 - (7 - 3x_1 + w_2 - x_0) \\ \hline w_1 = & 2 - 2x_1 + w_2 \\ x_2 = & 7 - 3x_1 + w_2 - \qquad\qquad\qquad x_0 \\ w_3 = & 1 - 2x_1 + w_2 + (7 - 3x_1 + w_2 - x_0) \end{array} \quad \rightsquigarrow \quad \begin{array}{rl} \xi = & 0 \qquad\qquad\qquad + x_0 \\ \hline w_1 = & 2 - 2x_1 + \ \ w_2 \\ x_2 = & 7 - 3x_1 + \ \ w_2 - x_0 \\ w_3 = & 8 - 5x_1 + 2w_2 - x_0 \end{array}$$

The phase one problem is solved once the objective value reaches 0, which typically happens exactly when x_0 leaves the basis. Once this happens, we can solve the **phase two problem**: the one we started with! To get there, we:

1. Remove x_0 from the dictionary; we no longer need it.
2. Replace the artificial objective function ξ by the original objective function ζ, expressed in terms of the current basic variables.

In this case our original objective is to minimize $\zeta = 4.5x_1 + 3x_2$. Substituting $x_2 = 7 - 3x_1 + w_2$ gives us $\zeta = 21 - 4.5x_1 - 3w_2$, so our new dictionary is:

$$\begin{array}{rl} \zeta = & 21 - 4.5x_1 + 3w_2 \\ \hline w_1 = & \ \ 2 - \ \ \ 2x_1 + \ \ w_2 \\ x_2 = & \ \ 7 - \ \ \ 3x_1 + \ \ w_2 \\ w_3 = & \ \ 8 - \ \ \ 5x_1 + 2w_2 \end{array}$$

Since we are minimizing ζ, the only good choice of entering variable is x_1. Comparing the ratios $\frac{2}{2}$, $\frac{7}{3}$, and $\frac{8}{5}$, we see that w_1 must leave the basis. Solving w_1's equation for x_1, we get $x_1 = 1 - \frac{1}{2}w_1 + \frac{1}{2}w_2$. Now we substitute that into the other equations:

$$\begin{array}{rcl} \multicolumn{3}{c}{\zeta = 21 - 4.5(1 - \frac{1}{2}w_1 + \frac{1}{2}w_2) + 3w_2} \\ \hline x_1 = & 1 - & \frac{1}{2}w_1 + \frac{1}{2}w_2 \\ x_2 = & 7 - & 3(1 - \frac{1}{2}w_1 + \frac{1}{2}w_2) + w_2 \\ w_3 = & 8 - & 5(1 - \frac{1}{2}w_1 + \frac{1}{2}w_2) + 2w_2 \end{array} \quad \rightsquigarrow \quad \begin{array}{rrrr} \multicolumn{4}{c}{\zeta = 16.5 + 2.25w_1 + 0.75w_2} \\ \hline x_1 = & 1 - & \frac{1}{2}w_1 + & \frac{1}{2}w_2 \\ x_2 = & 4 + & \frac{3}{2}w_1 - & \frac{1}{2}w_2 \\ w_3 = & 3 + & \frac{5}{2}w_1 - & \frac{1}{2}w_2 \end{array}$$

Since we are minimizing ζ and all our reduced costs are nonnegative, we have found the optimal solution. With 1 cup of fancy coffee and 4 cups of sweet tea (exceeding our sugar minimum by 3 units), we have found the cheapest combination of drinks, costing \$16.50.

3 The general two-phase simplex method

Suppose we have a general linear program in equational form: our constraints are written as $A\mathbf{x} = \mathbf{b}$, with $\mathbf{x} \geq \mathbf{0}$. Our approach in the previous section relied on having inequality constraints, so it no longer applies.

One silver lining is that we can always make the right-hand side nonnegative. An equation constraint can always be multiplied by -1 and remain valid (unlike an inequality constraint, which reverses when it is multiplied by -1). So let's assume that $\mathbf{b} \geq \mathbf{0}$.

The solution here is to introduce **artificial slack variables** to the problem. We turn the problem $A\mathbf{x} = \mathbf{b}$ into the problem $A\mathbf{x} \leq \mathbf{b}$, and then add slack variables $w_1, w_2, \dots, w_m \geq 0$ to turn it *back* into equational form. (In matrix form, this looks like $A\mathbf{x} + I\mathbf{w} = \mathbf{b}$.)

What's the point? Well, because we've assumed $\mathbf{b} \geq \mathbf{0}$, the new is a problem for which the two-phase simplex method is not necessary: if we make the slack variables $w_1, w_2, \dots, w_m$ our basic variables, we get an initial basic feasible solution.

As before, we introduce an artificial objective function to optimize in the phase one problem. In this case, our slack variables $w_1, w_2, \dots, w_m$ are artificial: they do not belong in the problem, since we want to have $A\mathbf{x} = \mathbf{b}$ and not just $A\mathbf{x} \leq \mathbf{b}$. So we decide to minimize $\xi = w_1 + w_2 + \cdots + w_m$: the sum of the slack variables. If we can get it down to $\mathbf{0}$, then we get a solution where $A\mathbf{x} = \mathbf{b}$, and then we can proceed to the phase two problem.

For example, suppose that we have the following constraints:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 6x_1 - 2x_3 = 1 \\ 2x_1 + x_2 - 3x_3 = -1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Our first step is to rewrite the third constraint as $-2x_1 - x_2 + 3x_3 = 1$, so that all the numbers on the right-hand side are positive. Now we are ready to insert artificial slack variables w_1, w_2, w_3:

$$\begin{cases} x_1 + x_2 + x_3 + w_1 & = 1 \\ 6x_1 - 2x_3 + w_2 & = 1 \\ -2x_1 - x_2 + 3x_3 + w_3 & = 1 \\ \phantom{-2x_1 - x_2 +{}} x_1, x_2, x_3, w_1, w_2, w_3 & \geq 0 \end{cases}$$

Our objective function for the phase one problem is $\xi = w_1 + w_2 + w_3$, but that's phrased entirely in terms of the basic variables. We must substitute $w_1 = 1 - x_1 - x_2 - x_3$, $w_2 = 1 - 6x_1 + 2x_3$, and $w_3 = 1 + 2x_1 + x_2 - 3x_3$ to get the objective function in the form we want. If we do, then ξ simplifies to $3 - 5x_1 - 2x_3$, and we get the initial dictionary

$$\begin{array}{rcl} \xi & = & 3 - 5x_1 \phantom{{}- x_2} - 2x_3 \\ \hline w_1 & = & 1 - x_1 - x_2 - x_3 \\ w_2 & = & 1 - 6x_1 \phantom{{}- x_2} + 2x_3 \\ w_3 & = & 1 + 2x_1 + x_2 - 3x_3 \end{array}$$

As before, we will proceed to minimize ξ, hoping to get to 0.

4 Troubleshooting

There are several unexpected things that can go wrong in the two-phase simplex method.

It is possible that we can never get the artificial objective function ξ down to 0. This is an indicator that our original problem did not have a feasible solution! Although this is disappointing for the problem we were trying to solve, it's convenient for the solver: now we can skip phase two.

In most cases, we expect that ξ will hit 0 at the same time that our artificial variable(s) leave the basis. After all, if $\xi = 0$, then x_0 (from our first two-phase method) or the artificial slack variables $w_1, \dots, w_m$ (from our second method) must all be 0, which is the sort of thing nonbasic variables generally do. However, it is possible for these variables to be basic and still be equal to 0.

In such a degenerate case, we can make some quick final adjustments. If an artificial variable is equal to 0 but still basic, pick any nonbasic, non-artificial variable in its equation, and do a pivot step to replace the artificial variable by that nonbasic variable—ignoring our usual pivoting rules. Because both variables will remain equal to 0, this will not change the value of any other variables, so this pivot step preserves feasibility.

In our second two-phase method, an even weirder thing can happen. It's possible that:

- The artificial objective function ξ has reached 0;
- Some artificial slack variable w_i is still basic;
- There are *no* non-artificial variables in w_i's equation to replace it with!

If this happens, just forget that equation entirely. What this means is that one of the equations in the system $A\mathbf{x} = \mathbf{b}$ was redundant; it could be deduced from the others. Once we eliminate the artificial slack variables, the redundant equation becomes $0 = 0$; we don't need it.