Math 2390 Lecture 19: Peano's Axioms

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1 Introduction

We've already seen this semester that proofs vary a lot depending on context. We always want to prove a new theorem in terms of facts we already know, but "facts we already know" can vary. After we've written a proof that describes the key steps leading to a theorem, we can often add explanations that justify those key steps. Those explanations have steps of their own that we can try to justify further and so forth. At some point, we have to have some facts that we assume without proof.

Today, we will dig a lot deeper than we usually do, going back to the very basics of what the natural numbers are. We will prove some things that we usually take for granted when dealing with numbers, just to see how those things can be justified. Along the way, we will see our first glimpse of induction.

Do your best to forget everything you know about the natural numbers for today: we are not going to assume anything beyond what we say about them today.

2 The natural numbers

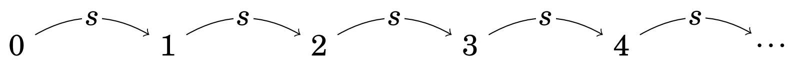
Just for today, we want the natural numbers to start with 0: it is traditional to do so when looking at these basic facts. To help you remember this, I will write $\mathbb{N}_0$ for the set $\{0,1,2,3,\dots\}$: the natural numbers together with 0.

Informally, what is this set? It is the set of all the numbers we get by counting up from 0. This idea of "counting up" will guide us to some axioms for how the natural numbers work.

The axioms we will be working with are called **Peano's axioms**. They have three built-in, fundamental notions. The first of them is the set $\mathbb{N}_0$, which we don't know anything about at all yet. The second notion is **zero**, or 0: an element of $\mathbb{N}_0$. The third is the **successor function**, a function $s\colon \mathbb{N}_0 \to \mathbb{N}_0$. The successor function formalizes the idea of "counting up": if $n \in \mathbb{N}_0$ is any number, then its **successor** $s(n)$ is the "next" number.

Of course, much as we'd like to, we're not coming into this blind, and so we have some of our own ideas about how we expect these fundamental notions to work.

Here is the picture of $\mathbb{N}_0$ that we expect to see:



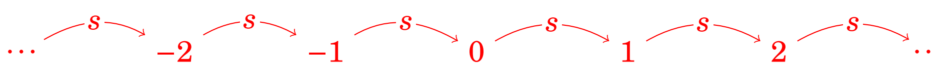
It's important, however, to keep two separate thoughts in your head: what we expect to see, and what our axioms tell us we must see.

What axioms?

The first two of Peano's axioms are:

1. There is no $n \in \mathbb{N}_0$ such that $s(n) = 0$.

   This axiom rules out the following picture (which I will draw in red because it's *not* what we expect from the natural numbers):

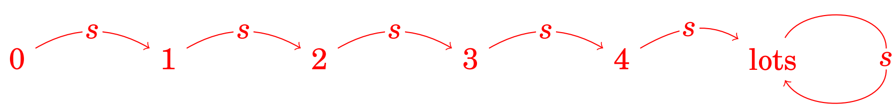
   

   Instead, the axiom tells us that the natural numbers start at 0.

2. For all $m, n \in \mathbb{N}_0$, if $s(m) = s(n)$, then $m = n$.

   (Later this semester, we will learn to say in such cases that s is **injective**.)

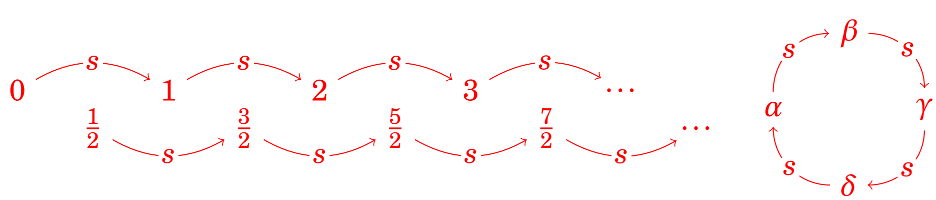
   This axiom makes sure that as we count upwards, we keep getting *new* numbers, unlike the picture below:

   By counting upward, we should be getting new numbers every time.

It feels as though we've covered everything; it's hard to imagine a picture of $\mathbb{N}_0$ in our heads that satisfies both of the axioms, but does something different from what we expect. However, there *are* such pictures.

The following diagram also satisfies all our axioms so far:



This diagram *contains* the picture we expect from the natural numbers, but also includes some extra details: it includes numbers like $\frac{1}{2}$ that we *don't* want to be natural numbers, and even some weird objects like α and β that we don't recognize as a form of counting at all! We want to add an axiom that rules this sort of nonsense out—that says that the picture in our heads of what the natural numbers should look like is *all* that there is.

It's tempting, at this point, to try adding the following as an axiom: "$\mathbb{N}_0$ only contains elements that we can get to by counting up some number of times from 0". The problem with this sentence is that we don't know what "some number of times" means before we define what natural numbers are! Suppose I say, "Well, the number α I drew in my picture is part of that definition; it's the number you get by counting up α times from 0." This is nonsense, but is it nonsense that the axiom rules out?

Instead, we add the following axiom:

3. If a set $X \subseteq \mathbb{N}_0$ contains 0, and for all $n \in X$, we also have $s(n) \in X$, then $X = \mathbb{N}_0$.

This axiom rules out the most recent false picture I drew. In that picture, let X be the set of numbers in the first chain, the one we expect from the natural numbers. This satisfies both conditions: $0 \in X$, and for all $n \in X$, we also have $s(n) \in X$. However, X does not include numbers like $\frac{1}{2}$ or α: it is not all of what $\mathbb{N}_0$ is according to that picture. Therefore the picture does not satisfy the axiom.

It turns out that with these three axioms, there is essentially only one possible picture of the natural numbers, and it is the picture we expect to see. We will not prove that—instead, we will go on to see what can be done with these axioms.

3 Definitions and proofs

The axioms stop before talking about any properties of natural numbers that we care about. This is because we can define everything else ourselves. Some definitions will just add simpler ways to say things we could already say; for other definitions, we'll need to prove that they work the way we want them to.

For example, right now, the only names we have for any natural numbers other than 0 are "$s(0)$", "$s(s(0))$", "$s(s(s(0)))$", and so forth. Purely to make our life easier, we can start using the usual names like "1", "2", "3", and so forth, as long as we understand that as far as the axioms are concerned, all that the symbol 5 actually means is "$s(s(s(s(s(0)))))$".

Here's another example: calling numbers "even" or "odd". Just from the axioms, we can try to write down a definition:[1]

Definition 1. *A number $n \in \mathbb{N}_0$ is called* ***odd*** *when $n = s(m)$ for some even natural number m.*

A number $n \in \mathbb{N}_0$ is called ***even*** *when $n = 0$ or when $n = s(m)$ for some odd natural number m.*

[1] Note that it is a different definition from the one we usually use in this class: that an integer n is even if there is an integer k such that $n = 2k$, and odd if there is an integer k such that $n = 2k+1$. There is a relationship between these definitions, but it would have to be proven before we use it. After this lecture, we will return to our usual definition—this one is just given as an example of how to use Peano's axioms.

Making a definition can be freely done no matter what, but checking that the definition does what we expect requires proof! For this definition, we can prove the following:

Theorem 1. *Every natural number is even or odd, but not both.*

Proof. We will prove two separate claims: first, that every $n \in \mathbb{N}_0$ is either even or odd; second, that no $n \in \mathbb{N}_0$ is both. Both of them will rely heavily on our third axiom: it's the one that lets us say things about *every* natural number!

For the first claim, let X be the set of all natural numbers that are either even or odd. Then $0 \in X$, because by definition 0 is even. Suppose that $n \in X$. Then there are two cases:

- n is even; in this case, $s(n)$ is odd (by definition), so $s(n) \in X$.
- n is odd; in this case, $s(n)$ is even (by definition), so $s(n) \in X$.

We've checked that X satisfies both conditions of axiom 3, so $X = \mathbb{N}_0$. Therefore all natural numbers are either even or odd!

For the second claim, define X differently: let it be the set of all natural numbers that are not both even or odd. Why is $0 \in X$? Because 0 is not odd: there is no even natural number m such that $0 = s(m)$, because (by axiom 1) there is no such natural number at all.

Now, to check the second condition in axiom 3, suppose $n \in X$. We will prove that $s(n) \in X$ by contradiction: suppose that $s(n) \notin X$, so that $s(n)$ is both even and odd. Then two things must be true:

1. Because $s(n)$ is odd, there is some even natural number m such that $s(n) = s(m)$. By axiom 2, $n = m$. Therefore n is even.
2. Because $s(n)$ is even, either $n = 0$ (which is ruled out by axiom 1) or there is some odd natural number m such that $s(n) = s(m)$. By axioms 2, $n = m$. Therefore n is odd.

The result is that n is also both even and odd, contradicting the assumption that $n \in X$.

As a result, X satisfies both conditions of axiom 3, so $X = \mathbb{N}_0$. Therefore there is no natural number that is both even or odd. □

(Note: we could have proved this theorem more "efficiently" in one step, by defining X to be the set of natural numbers that are either even or odd, but not both. Checking the conditions of axiom 3 would be more work, but then we'd be done as soon as we did it.)

We can go further and define addition of natural numbers. Working from Peano's axioms, pretty much every definition has to be done recursively (or with the help of previous definitions, which we don't have many of yet). So we can say:

Definition 2. *For all $m, n \in \mathbb{N}_0$, $m + n$ is the natural number given by*

$$m + n = \begin{cases} m & \text{if } n = 0, \\ s(m+k) & \text{if } n = s(k). \end{cases}$$

Before we do anything else, how do we use this definition? Here's how:

Theorem 2. $2+2=4$.

Proof. To evaluate $2+n$, where $n=2$, we need to check which case of the piecewise definition we're in. It is not true that $2=0$ (formally, because $2=s(s(0))$ and by axiom 1, 0 is not the successor of anything). It is, however, true that $2=s(1)$. Therefore we take the second case of the definition, and have $2+2=s(2+1)$.

We apply the same reasoning to find $2+1$. Again, $1\neq 0$, so we're not in the first case; however, the second case does apply, since $1=s(0)$. Therefore $2+1=s(2+0)$, which we can substitute into our previous equation to get $2+2=s(s(2+0))$.

Finally, we get down to $2+0$. Here, we're finally in the first case of the equation, so $2+0$ is just 2. Therefore $2+2=s(s(2))=s(3)=4$. □

Okay, but actually, before we proved that theorem, it would have been good to check that our definition of $m+n$ is a valid definition that makes sense.

This is a concern because a binary operation like $m+n$ is really a function $\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{N}_0$, but we're not defining $m+n$ by directly giving a formula for what $m+n$ is for every pair $(m,n)\in\mathbb{N}_0\times\mathbb{N}_0$. Instead, we're saying that the different values of $m+n$ satisfy a certain relationship: the recurrence relation in our definition. So there's two things to be worried about:

1. Maybe there is no consistent way to define $m+n$ for every pair $(m,n)\in\mathbb{N}_0\times\mathbb{N}_0$ at all. Maybe if you go out and try to fill in the addition table for $0+0, 0+1, \dots, 9+9$, then no matter how you do it, the relationship in the definition will sometimes fail to work.

   (In this case, the theorem that $2+2=4$ would be a theorem about a nonsense function that doesn't even exist.)

2. Maybe there are two operations, $*$ and $\oplus$, that both satisfy the recurrence relation, so that

$$m*n=\begin{cases} m & \text{if } n=0,\\ s(m*k) & \text{if } n=s(k),\end{cases}\qquad\text{and}\qquad m\oplus n=\begin{cases} m & \text{if } n=0,\\ s(m\oplus k) & \text{if } n=s(k),\end{cases}$$

   but for some values of m and n, $m*n\neq m\oplus n$. Then, for those values of m and n, writing $m+n$ would be ambiguous: there's multiple possible values for $m+n$, because we don't know if $+$ means $*$ or $\oplus$.

   (For the theorem we proved, this is less of a concern: if there were two different possible operations, what the theorem would prove is that $2*2$ and $2\oplus 2$ must both equal 4, even if they disagree elsewhere. But it's otherwise something we would very much like to know.)

The proof of these facts is a lot like our proof that every natural number is even or odd, but not both. For example, in the hypothetical situation that $*$ and $\oplus$ both exist, we could pick an arbitrary m to use on the left side of the operation, and let $X = \{n \in \mathbb{N}_0 : m * n = m \oplus n\}$. Then, if we check that the conditions of axiom 3 apply, it would tell us that $x = \mathbb{N}_0$: that $m * n = m \oplus n$ for all n. Since m was also arbitrary to begin with, we'd conclude that the two operations always agree!

In fact, a general theorem is true, and can we prove in this way: if we want to define a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ by giving a value $f(0) = a$ and a rule that gives $f(s(n))$ in terms of $f(n)$ and n, then this always successfully defines a unique function.

Of course, beyond just showing that $+$ exists, we might want to prove various properties of $+$.

Theorem 3. *For all $n \in \mathbb{N}_0$, $0 + n = n + 0 = n$.*

Proof. Of the two equations in this theorem, one doesn't need much work at all: we have $n + 0 = n$ by definition. To show $0 + n = n$, we need a bit more work.

Let $X = \{n \in \mathbb{N}_0 : 0 + n = n\}$. We have $0 \in X$, because $0 + 0$ is defined directly to be 0.

Suppose $n \in X$. Then $0 + s(n)$ falls under the second case of the piecewise definition of $+$, it is $s(0+n)$. Because $n \in X$, we know that $0+n = n$, so $0+s(n) = s(0+n) = s(n)$. Therefore $s(n) \in X$ as well.

Now, by axiom 3, $X = \mathbb{N}_0$, so $0 + n = n$ for all $n \in \mathbb{N}_0$. □

4 Where to go from here

More can be done—for one thing, we haven't even *defined* multiplication. Since our class is not primarily about the Peano axioms and building number theory from the ground up, we will abandon that train of thought and go back to our usual way of thinking about the natural numbers after today's glimpse at its foundations.

We will keep one thing: axiom 3. This axiom is called the **axiom of induction**, and it is a very powerful tool that we can keep using even after we're far past proving basic statements like "$0+n = n$". In the next few lectures, we will see many other ways that it can be used.

5 References

These notes are written so that you have a self-contained writeup of everything we cover regarding Peano's axioms. They are heavily based on section 4.1 of Clive Newstead's textbook *An Infinite Descent into Pure Mathematics*, which is available online from https://infinitedescent.xyz/. Read that textbook if you want to learn more!