Graph Theory Homework 6

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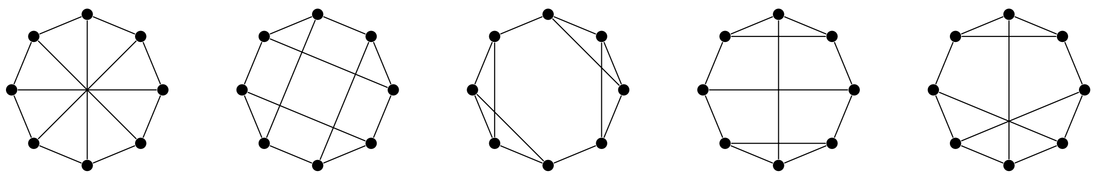
due Friday, April 8, 2022

1 Short answer

1. Here are the five connected 3-regular graphs on 8 vertices.



   Three of these are planar, and two are not.

   (a) Find which three of these graphs are planar, and draw a plane embedding of each.

   (b) In the two remaining graphs, find a subdivision of $K_{3,3}$ to show that they are not planar. It has to be $K_{3,3}$, because a 3-regular graph can't possibly have a subdivision of K_5.

   (Hint: we can draw $K_{3,3}$ as a regular hexagon with opposite corners connected.)

2. A planar graph has 15 vertices. Two of its faces have length 6, and all remaining faces are triangles (they have length 3).

   (a) Determine the number of triangular faces.

   (b) If all the vertices in the graph have degree 4 or 5, how many of each degree are there?

   (As usual, I am not requiring you to show work, but if you make a mistake, showing enough work that I can find the mistake you made is the only way you can get partial credit.)

3. We know that the complete bipartite graph $K_{3,3}$ is not planar. However, for all n, the complete bipartite graph $K_{2,n}$ *is* planar.

   (a) Draw a plane embedding of $K_{2,n}$ where n is some very big number—like 6. Explain how to keep going and draw such a plane embedding for larger values of n.

   (b) Find the dual graph of the plane embedding you drew in the first part. Explain how the dual graph will look for larger values of n.

   (When you're working out the answer, it might help to draw the dual graph on top of the plane embedding, but please make the two drawings separate for your final answers.)

2 Proof

4. Prove that every sequence $d_1, d_2, \ldots, d_n$ such that $d_1 + d_2 + \cdots + d_n$ is even is the degree sequence of some multigraph. (Don't forget that a loop contributes 2 to the degree of a vertex.)

   You have already written a rough draft of this problem. Now, write the final draft.

5. Do **one** of the following; for extra credit, do both.

   (a) Prove that for all $n \geq 6$, if a connected n-vertex graph has $n + 2$ edges or fewer, then it is planar.

   (b) Prove that for all $n \geq 6$, there is a connected n-vertex graph with $n + 3$ edges which is not planar.

   Write a rough draft of the solution. I will give you feedback, and you will write a final draft of your proof as part of Homework 6.