Math 2390 Homework 3

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1 Short answer

1. Write the statements below formally, using quantifiers and mathematical notation.

   (a) Every real solution of the equation $e^x = \sin x$ is negative.

   (b) Between any two different rational numbers, there is a third rational number.

2. The statement "97 is a prime number" can be expanded to "If a and b are natural numbers with $ab = 97$, then $a = 1$ or $b = 1$."

   (a) How would you begin a direct proof of this statement?

   (b) How would you begin a proof of this statement by contrapositive?

3. Here is a proof of the statement "There are arbitrarily large even integers." From the last homework assignment, we know that this can be expressed as "For all $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $n > m$ and n is even." Read the proof and answer the questions after it.

   Proof. Let m be an arbitrary integer.

   Case 1: m is odd. Then there is some $k \in \mathbb{Z}$ such that $m = 2k + 1$. Let $n = m + 1$. Then $n > m$ and n is even.

   Case 2: m is even. Then there is some $k \in \mathbb{Z}$ such that $m = 2k$. Let $n =$ ________. Then $n > m$ and n is even. □

   (a) The argument in Case 1 does not explain why n is even. How would you prove this carefully, from the definition?

   (b) How could we define n in Case 2 to make the proof work?

2 Proof

4. *For each part of this problem, write a "proof" with the structure of a real proof of the given statement, taking into account all the definitions and quantifiers given. When you get to the point where you'd need to have unicorn-, dragon-, or magic-related knowledge, make something up.*

(a) A unicorn is considered **huge** if it is at least as large as any other unicorn. Prove that all white unicorns are huge.

(b) A dragon is called **rich** if its hoard contains a diamond, and **clever** if its hoard contains a magic item. Prove that all clever dragons are rich.

5. *For this problem, write a rough draft of an actual proof. I will give you feedback, and you will write a final draft for Homework 4.*

   Prove that for all $a, b \in \mathbb{Z}$, if $a + b = 100$, then a and b are both even or both odd.