Math 2390 Homework 5

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due Friday, March 25, 2022

1 Short answer

1. Disprove the statement: for all natural numbers n, $n^2 + n + 11$ is prime.
2. Suppose we want to prove the statement: for all real a, b, if $a^2 + b^2 \leq 8$, then $ab \leq 4$.

   Classify each of the following initial assumptions as either a mistake, or the beginning of a direct proof, proof by contrapositive, or proof by contradiction.

   (a) Let $a, b \in \mathbb{R}$ be arbitrary, and assume that $ab > 4$.

   (b) Let $a, b \in \mathbb{R}$ be arbitrary, and assume that $a^2 + b^2 \leq 8$ and $ab \leq 4$.

   (c) Let $a, b \in \mathbb{R}$ be arbitrary, and assume that $ab \leq 4$.

   (d) Let $a, b \in \mathbb{R}$ be arbitrary, and assume that $a^2 + b^2 \leq 8$, but $ab > 4$.
3. Consider the following claim: "For all integers n, $n^2 + n + 1$ is not divisible by 17".

   (a) Is the claim a counterexample that disproves the assertion "For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $x^2 + y + 1$ is divisible by 17?" Why or why not?

   (b) Is the claim a counterexample that disproves the assertion "There exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $x^2 + y + 1$ is divisible by 17?" Why or why not?

2 Proof

4. *You have already written a rough draft of this proof; now, write a final draft.*

   Prove that for all sets A and B, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.
5. *For this problem, write a rough draft of an actual proof. I will give you feedback, and you will write a final draft for Homework 6.*

   Prove one of the following statements and disprove the other:

   - For all natural numbers n, if $n \equiv 2 \pmod 5$, then $n^2 - 2n$ is divisible by 5.
   - For all natural numbers n, if $n \equiv 2 \pmod 5$, then $n^2 + 2n + 2$ is divisible by 10.