Math 2390 Homework 6

Mikhail Lavrov

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1 Short answer

1. In class, we defined addition and multiplication over $\mathbb{N}_0$ as follows:

   - If $m \in \mathbb{N}_0$, we define a function $\text{add}_m \colon \mathbb{N}_0 \to \mathbb{N}_0$ recursively by $\text{add}_m(0) = m$ and $\text{add}_m(S(n)) = S(\text{add}_m(n))$ for $n \in \mathbb{N}_0$. The notation $m + n$ is shorthand for $\text{add}_m(n)$.
   - If $m \in \mathbb{N}_0$, we define a function $\text{mult}_m \colon \mathbb{N}_0 \to \mathbb{N}_0$ recursively by $\text{mult}_m(0) = 0$ and $\text{mult}_m(S(n)) = \text{mult}_m(n) + m$ for $n \in \mathbb{N}_0$. The notation $m \cdot n$ is shorthand for $\text{mult}_m(n)$.

   Use the recursive definitions of addition and multiplication to demonstrate each of the following, step by step.

   (a) $1 + 3 = 4$.

   (b) $3 \cdot 1 = 3$. You may simplify addition in this part without justifying it.

2. Suppose you are trying to use induction to prove that

$$\frac{4}{1 \cdot 3} + \frac{4}{2 \cdot 4} + \frac{4}{3 \cdot 5} + \dots + \frac{4}{n(n+2)} = \frac{n(3n+5)}{(n+1)(n+2)}.$$

   What identity will you need to check in the induction step?

   (You don't actually need to prove anything or check any identities to answer this problem: just tell me what you'd need to prove.)

3. A sequence q_n is defined by $q_1 = 1$ and the recurrence relation $q_{n+1} = 2q_n + 2^n$ which holds for all positive integers n.

   A student is proving a closed-form expression for q_n by induction, and has the following step in their work:

$$2 \cdot n \cdot 2^{n-1} + 2^n = n \cdot 2^n + 2^n = (n+1) \cdot 2^n.$$

   (a) What is the formula for q_n that the student is trying to prove?

   (b) What else must be checked to know that the formula is correct?

2 Proof

4. *You have already written a rough draft of this proof; now, write a final draft.*

   Prove one of the following statements and disprove the other:

   - For all natural numbers n, if $n \equiv 2 \pmod{5}$, then $n^2 - 2n$ is divisible by 5.
   - For all natural numbers n, if $n \equiv 2 \pmod{5}$, then $n^2 + 2n + 2$ is divisible by 10.

5. *For this problem, write a rough draft of an actual proof. I will give you feedback, and you will write a final draft for Homework 7.*

   Let a_n be the sum $2^{0^2} + 2^{1^2} + 2^{2^2} + \cdots + 2^{n^2}$. For example, $a_4 = 2^0 + 2^1 + 2^4 + 2^9 + 2^{16}$.

   Prove by induction on n that for all $n \in \mathbb{N}$, a_n is odd.